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Commuting planar polynomial vector fields for conservative Newton systems*

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1 Introduction

We study the problem of characterizing polynomial vector fields that commute with a given polynomial vector field on a plane. It is a classical result that one can write down solution formulas for an ODE that corresponds to a planar vector field that possesses a linearly independent (transversal) commuting vector field (see Theorem 2.1). In what follows, we will use the standard correspondence between (polynomial) vector fields and derivations on (polynomial) rings. Let

$$d = y\frac{\partial}{\partial x} + f(x)\frac{\partial}{\partial y} \tag{1}$$

be a derivation, where f is a polynomial with coefficients in a field K of zero characteristic. This derivation corresponds to the differential equation $\ddot{x} = f(x)$, which is called a conservative Newton system as it is the expression of Newton's second law for a particle confined to a line under the influence of a conservative force. Let H be the Hamiltonian polynomial for d with zero constant term, that is $H = \frac{1}{2}y^2 - \int_0^x f(t)dt$. Then the set of all polynomial derivations that commute with d forms a $K[H]$-module M_d [6, Corollary 7.1.5]. We show that, for every such d, the module M_d is of rank 1 if and only if $\deg f \geqslant 2$. For example, the classical elliptic equation $\ddot{x} = 6x^2 + a$, where $a \in \mathbb{C}$, falls into this category. For proofs of the results stated in this abstract, see [5].

In the case in which K is the real numbers, our result generalizes a result on conservative Newton systems with a center to the case in which a center may or may not be present. A vector field has a center at point P if there is a punctured neighborhood of P in which every solution curve is a closed loop. A center

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is called isochronous if every such loop has the same period. It was proven by Villarini [8, Theorem 4.5] that, if D_1 and D_2 are commuting vector fields orthogonal at noncritical points, then any center of D_1 is isochronous. The hypothesis of this result can be relaxed to the case in which D_2 is transversal to D_1 at noncritical points (cf. [7, Theorem, p. 92]). In light of this result, one approach to showing the nonexistence of a vector field commuting with D is to show that D has a non-isochronous center. In fact, Amel'kin [1, Theorem 11] has shown that, if the system of ordinary differential equations (ODEs) corresponding to derivation (1) is not linear and has a center at the origin, then there is no transversal vector field that commutes with d.

As far as we are aware, there has not been a standard method to show the nonexistence of a transversal polynomial vector field in the absence of a nonisochronous center. We develop our own method to do this. Write $\delta = y\frac{\partial}{\partial x} + f(x)\frac{\partial}{\partial y}$ and $\gamma = \sum_{i=0}^{M} c_i y^i \frac{\partial}{\partial x} + \sum_{i=0}^{M} d_i y^i \frac{\partial}{\partial y}$, with $c_i, d_i \in K[x]$ treated as unknowns. Now writing the two components of $[\delta, \gamma]$ as polynomials in y and setting each coefficient equal to 0 yields a system of linear equations in the c_i and d_i and their first derivatives. These equations have a different form depending on whether M is even or odd, and in either case the system is the disjoint union of two independent systems. Thus, we have four linear systems to address. Each system has the following triangularity property. The number of unknowns exceeds the number of equations by one. Label the system $e_0, \ldots, e_{m+1}$ and note that $m+1$ variables appear in the system. Only one variable appears in e_{m+1} and we can solve for it up to addition of a constant by antidifferentiation. The equation e_m contains the variable from e_{m+1} and one new variable, and we can solve for this up to addition of a constant using our solution to e_{m+1} and antidifferentiation. We continue this way, eventually solving e_1 for the final unknown variable, up to addition of a constant. What remains is to determine what choices of constants, if any, will yield a solution to e_0. In each case, we show that either no such choice of constants exists, or any choice of constants solving e_0 corresponds, in a sense we do not detail here, to an element of $K[H]\delta$. For this, one technique we use involves showing that, if the system has a solution with components in $K[x]$, then $\deg f$ must belong to a particular finite subset of $\mathbb{Q}$. Then we prove that this finite set does not intersect $\mathbb{Z}_{\geqslant 2}$. To do this, we construct a family of pairs of commuting derivations on rings of the form $K[x^{1/t}, x^{-1/t}, y]$.

It is impossible to remove the condition $\deg f \geqslant 2$ from the statement of our main result, as every non-zero derivation defined by polynomials of degree less than 2 commutes with another transversal derivation (see Proposition 2.1). The form of d in our main result implies that d is divergence free (which is the same as Hamiltonian in the planar case). It is not possible to strengthen our result to the case in which d is merely assumed to be divergence free of degree at least 2, as shown in Example 2.1 and Proposition 2.2.

2 Basic terminology and related results

Definition 2.1. An S-derivation on a commutative ring R with subring S is a map $d\colon R \to R$ such that $d(S) = 0$ and for all a, $b \in R$,

$$d(a+b) = d(a) + d(b) \quad \text{and} \quad d(ab) = d(a)\cdot b + a \cdot d(b).$$

Definition 2.2. Let K be a field. A non-zero K-derivation d on $K[x_1, \ldots, x_n]$ is called *integrable* if there exist commuting K-derivations $\delta_1, \ldots, \delta_{n-1}$ on $K[x_1, \ldots, x_n]$ that are linearly independent from d over $K(x_1, \ldots, x_n)$, and commute with d, that is, for all $a \in K[x_1, \ldots, x_n]$ and i,j, $1 \leqslant i, j \leqslant n-1$,

$$d(\delta_i(a)) = \delta_i(d(a)) \quad \text{and} \quad \delta_i(\delta_j(a)) = \delta_j(\delta_i(a)).$$

It is a known result (cf. [4, Theorem 9.46]) that, if a smooth vector field v on $\mathbb{R}^2$ admits a transversal commuting smooth vector field w at a point (x_0, y_0), then there is a coordinate change φ such that $\varphi(x_0, y_0) = (0, 0)$ and $\varphi_* v = (1, 0)$. Applying φ^{-1} to the integral curve $(t, 0)$ of $\varphi_* v$ yields an integral curve of v through (x_0, y_0). We state a more detailed version of this for the case of polynomial vector fields:

Theorem 2.1. *Let d and δ be $\mathbb{R}$-derivations on $\mathbb{R}[x,y]$ defined by*

$$d(x) = f_1(x,y),\ d(y) = f_2(x,y),\ \delta(x) = g_1(x,y),\ \delta(y) = g_2(x,y).$$

Let $(x_0, y_0) \in \mathbb{R}^2$. Suppose that d and δ commute and there is no $(\lambda_1, \lambda_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$ such that

$$\lambda_1 \begin{pmatrix} f_1(x_0,y_0) \\ f_2(x_0,y_0) \end{pmatrix} = \lambda_2 \begin{pmatrix} g_1(x_0,y_0) \\ g_2(x_0,y_0) \end{pmatrix}.$$

Then the initial value problem

$$\dot{x} = f_1(x,y),\ \dot{y} = f_2(x,y),\ x(0) = x_0,\ y(0) = y_0$$

has a solution given by

$$(x(t), y(t)) = F^{-1}(t, 0),$$

where

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \int\limits_{x_0}^{x} \frac{g_2(r,y)}{\Delta(r,y)} dr + \int\limits_{y_0}^{y} \frac{-g_1(x_0,s)}{\Delta(x_0,s)} ds \\ \int\limits_{x_0}^{x} \frac{-f_2(r,y)}{\Delta(r,y)} dr + \int\limits_{y_0}^{y} \frac{f_1(x_0,s)}{\Delta(x_0,s)} ds \end{pmatrix}, \text{ and } \Delta(x,y) = \begin{vmatrix} f_1(x,y) & g_1(x,y) \\ f_2(x,y) & g_2(x,y) \end{vmatrix}.$$

Proof. Suppose $(x(t), y(t))$ is a solution to the initial value problem. A straightforward calculation shows that $F(x(t), y(t)) = (t, 0)$. Observing that the Jacobian determinant of F does not vanish at (x_0, y_0), we see that F is a diffeomorphism in a neighborhood of (x_0, y_0). We conclude that $(x(t), y(t)) = F^{-1}(t, 0)$. □

Example 2.1. Consider the initial value problem

$$\dot{x} = 1 + x^2, \quad \dot{y} = -2xy, \quad x(0) = x_0, \quad y(0) = y_0,$$

where x_0 and y_0 are real numbers and $y_0 \neq 0$. The corresponding derivation is

$$d(x) = 1 + x^2, \quad d(y) = -2xy,$$

and we observe that the derivation

$$\delta(x) = 0, \quad \delta(y) = y$$

commutes with d, and that d and δ are independent at (x_0, y_0). Using the above formula, we obtain the solution $x(t) = \tan(t + \tan^{-1} x_0)$, $\quad y(t) = y_0(1 + x_0^2)\cos^2(t + \tan^{-1} x_0)$.

We make some observations, in the form of the following propositions:

Proposition 2.1. *Let K be a field. Every non-zero K-derivation on $K[x,y]$ defined by polynomials of degree less than or equal to 1 is integrable.*

Definition 2.3. Let K be a field and d be a K-derivation on $K[x_1, \ldots, x_n]$. We say d is *divergence-free* if

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} d(x_i) = 0.$$

Proposition 2.2. *Let K be a field of characteristic 0. There exist integrable divergence-free K-derivations on $K[x,y]$ that are not equivalent to a derivation of degree less than or equal to 1 under any K-automorphism of $K[x,y]$.*

In the following section, we study a class of divergence-free vector fields. We show that no member of this class is integrable.

3 Main result

Fix a field K of characteristic 0. Suppose δ_f represents a second-order differential equation of the form $\ddot{x} = f$, where $f \in K[x] \setminus K$, which corresponds to a conservative Newton system. That is,

$$\delta_f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ f \end{pmatrix} \tag{2}$$

If $\deg f = 1$, then δ_f is integrable by Proposition 2.1. The following theorem, which is our main result, addresses the case of $\deg f \geqslant 2$.

Theorem 3.1. *For every*

- $f \in K[x]$ *such that* $\deg f \geqslant 2$ *and*
- K*-derivation* γ *on* $K[x, y]$ *that commutes with* δ_f*, where* δ_f *is the* K*-derivation defined by* (2),

there exists $q \in K[H]$ *such that* $\gamma = q \cdot \delta_f$*, where* $H = y^2 - 2\int f dx$ *and* $\int f dx$ *has* 0 *as the constant term.*

As a corollary, we recover the following result on conservative Newton systems with a center at the origin. This result was first proven in [1, Theorem 11] and was given new proofs in [2, Theorem 4.1] and [3, Corollary 2.6] (see also [9, p. 30]).

Corollary 3.1. *The real system* $\{\dot{x} = -y,\ \dot{y} = f(x)\}$ *with* $f(0) = 0$*,* $f'(0) = 1$*, has a transversal commuting polynomial derivation if and only if* $f(x) = x$*.*

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