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Offsets to conics and quadrics: a new determinantal representation for their implicit equation

Jorge Caravantes
Universidad Complutense de Madrid, Spain
jorgecaravan@gmail.com

Mario Fioravanti
Universidad de Cantabria, Spain
mario.fioravanti@unican.es

Laureano Gonzalez–Vega
Universidad de Cantabria, Spain
laureano.gonzalez@unican.es

Gema Diaz–Toca
Universidad de Murcia, Spain
gemadiaz@um.es

Abstract

A new determinantal presentation of the implicit equation for offsets to non degenerate conics and quadrics is introduced which is specially well suited for intersection purposes.

1 Introduction

Let δ be a positive real number. Given a curve $\mathcal{C}$ in $\mathbb{R}^2$, the δ-offset to $\mathcal{C}$, $\mathcal{C}_\delta$, is the locus of the points in $\mathbb{R}^2$ which are at constant Euclidean distance δ from the initial curve $\mathcal{C}$ (along its normal line). Given a surface $\mathcal{S}$ in $\mathbb{R}^3$, the δ-offset to $\mathcal{S}$, $\mathcal{S}_\delta$, is the locus of the points in $\mathbb{R}^3$ which are at constant Euclidean distance δ from the initial surface $\mathcal{S}$ (along its normal line). Offsets have many important practical applications in Computer Aided Design, such as tool path generation, NC milling, design of thick curved surfaces and tolerance analysis (see, for example, [2, 4]).

If $\mathcal{C}$ is a rational curve or if $\mathcal{S}$ is a rational surface then its offset is an algebraic curve or surface, but it is generally not rational. In addition, the offset implicit equation typically has much higher degree than initial curve or surface, many terms and big coefficients. We introduce here a new presentation for the implicit equation of $\mathcal{C}_\delta$ and $\mathcal{S}_\delta$ when $\mathcal{C}$ is a non degenerate conic and $\mathcal{S}$ is a non degenerate quadric. They are of determinantal type, free of extraneous components and well suited for intersection purposes (which is the main motivation for deriving these determinantal equations).

2 Describing $\mathcal{C}_\delta$: the offset to distance δ of a curve $\mathcal{C}$ in $\mathbb{R}^2$.

When $\mathcal{C}$ is given by a rational parameterisation: $(x(t), y(t))$, $t \in \mathbb{R}$, with $x(t)$ and $y(t)$ quotients of polynomials in $\mathbb{R}[t]$, the previous purely geometric definition has two natural interpretations. The first one provides a parametrisation for the offset curve: a parametric representation of $\mathcal{C}_\delta$ is given by

$$(X(t), Y(t)) = \left(x(t) \pm \delta \frac{y'(t)}{\sqrt{x'(t)^2 + y'(t)^2}}, y(t) \mp \delta \frac{x'(t)}{\sqrt{x'(t)^2 + y'(t)^2}} \right) \tag{1}$$

where, for $a \geq 0$, $\sqrt{a}$ denotes the non-negative square root of a. This representation for $\mathcal{C}_\delta$ consists on two parametric curves (generally not rational) which correspond to what are usually called the exterior and interior offsets. The second one provides a characterisation of a point $(x, y) \in \mathbb{R}^2$ being on the offset curve

$\mathcal{C}_\delta$. If a point (x, y) is on $\mathcal{C}_\delta$ then there exists $t \in \mathbb{R}$ such that:

$$\begin{aligned} F(x, y; t) &= (x - x(t))^2 + (y - y(t))^2 - \delta^2 = 0, \\ G(x, y; t) &= x'(t)\,(x - x(t)) + y'(t)\,(y - y(t)) = 0. \end{aligned}$$

Both interpretations need to take into account the singularities of $\mathcal{C}$ when they exist and the extraneous solutions that appear depending on how the points on $\mathcal{C}_\delta$ are characterized.

The offset of a rational curve is an algebraic curve: an implicit equation of the δ-offset $\mathcal{C}_\delta$ can be obtained by taking the resultant of F and G in (2) with respect to t (once the denominators in F and G have been removed). The implicit equation of $\mathcal{C}_\delta$ is typically a factor of such a resultant and, when $x(t)$ and $y(t)$ are polynomials, the implicit equation of $\mathcal{C}_\delta$ agrees with the resultant if the second equation in (2) is divided by $\gcd(x'(t), y'(t))$. The degree of the implicit equation of $\mathcal{C}_\delta$ is much higher than the degree of the original curve and it is usually a very dense polynomial. If the curve $\mathcal{C}$ is presented by its implicit equation $f(x, y) = 0$ then the computation of the implicit equation of $\mathcal{C}_\delta$ requires to eliminate α and β from the equations:

$$f(\alpha, \beta) = 0, \quad (x - \alpha)^2 + (y - \beta)^2 - \delta^2 = 0, \quad (x - \alpha) f_y(\alpha, \beta) - (y - \beta) f_x(\alpha, \beta) = 0$$

and it is not easy to have a closed (and compact) formulae for the implicit equation of $\mathcal{C}_\delta$.

3 Describing $\mathcal{S}_\delta$: the offset to distance δ of a surface $\mathcal{S}$ in $\mathbb{R}^3$.

If the surface $\mathcal{S}$ is presented by a parameterization $(x(s, t), y(s, t), z(s, t))$ then then the computation of the implicit equation of $\mathcal{S}_\delta$ requires to eliminate s and t from the equations:

$$\begin{aligned} F(x, y, z; s, t) &= (x - x(s, t))^2 + (y - y(s, t))^2 + (z - z(s, t))^2 - \delta^2 = 0, \\ G(x, y, z; s, t) &= x_s(s, t)\,(x - x(s, t)) + y_s(s, t)\,(y - y(s, t)) + z_s(s, t)\,(z - z(s, t)) = 0 \\ H(x, y, z; s, t) &= x_t(s, t)\,(x - x(s, t)) + y_t(s, t)\,(y - y(s, t)) + z_t(s, t)\,(z - z(s, t)) = 0. \end{aligned}$$

If the surface $\mathcal{S}$ is presented by its implicit equation $f(x, y, z) = 0$ then the computation of the implicit equation of $\mathcal{S}_\delta$ requires to eliminate α, β and γ from the equations:

$$f(\alpha, \beta, \gamma) = 0, \quad (x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 - \delta^2 = 0, \quad \begin{array}{l} (x - \alpha) f_y(\alpha, \beta, \gamma) - (y - \beta) f_x(\alpha, \beta, \gamma) = 0 \\ (y - \beta) f_z(\alpha, \beta, \gamma) - (z - \gamma) f_y(\alpha, \beta, \gamma) = 0 \end{array}.$$

In both cases it is not easy to have a closed (and compact) formulae for the implicit equation of $\mathcal{S}_\delta$.

4 On the implicit equation for the δ-offset to a conic.

The equation of any conic $\mathcal{A}$ in $\mathbb{R}^2$ can be written as

$$a_{11}x^2 + a_{22}y^2 + 2a_{12}xy + 2a_{13}x + 2a_{23}y + a_{33} = 0$$

or in matricial form $\begin{pmatrix} x & y & 1 \end{pmatrix} A \begin{pmatrix} x & y & 1 \end{pmatrix}^T = 0$ where A is the symmetric matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}.$$

Given two conics $\mathcal{A} : XAX^T = 0$ and $\mathcal{B} : XBX^T = 0$, their characteristic equation is defined as $f(\lambda) = \det(B\lambda + A) = \det(B)\lambda^3 + \ldots + \det(A)$ which is, if $\det(B) \neq 0$, a cubic polynomial in λ with real coefficients.

We say that two non degenerate conics are touching each other externally if they are separated by a line tangent to both conics. Next theorem characterises when two conics touch each other externally in terms of the real roots of its characteristic equation (see [5]).

Theorem 1 *Let $\mathcal{A} : XAX^T = 0$ be a non degenerate conic, $\mathcal{B} : XBX^T = 0$ a circle and $f(\lambda) = \det(A + B\lambda)$ their characteristic equation. Then $\mathcal{A}$ and $\mathcal{B}$ touch each other externally if and only if $f(\lambda)$ has a positive double root.*

Let $\mathcal{A} : XAX^T = 0$ be a non degenerate conic and $\mathcal{B} : XBX^T = 0$ the circle with center in (x_c, y_c) and radius δ:

$$B = \begin{pmatrix} 1 & 0 & -x_c \\ 0 & 1 & -y_c \\ -x_c & -y_c & x_c^2 + y_c^2 - \delta^2 \end{pmatrix}.$$

The point (x_c, y_c) belongs to the (exterior) offset to $\mathcal{A}$ to distance δ if and only if $\mathcal{A}$ is externally tangent to $\mathcal{B}$. According to previous theorem, this happens if and only if the polynomial

$$f(\lambda) = \det(A + B\lambda) = \begin{vmatrix} \lambda + a_{11} & a_{12} & -\lambda x_c + a_{13} \\ a_{12} & \lambda + a_{22} & -\lambda y_c + a_{23} \\ -\lambda x_c + a_{13} & -\lambda y_c + a_{23} & \lambda\left(x_c^2 + y_c^2 - \delta^2\right) + a_{33} \end{vmatrix}$$

has a positive double root. But this implies the vanishing of the discriminant of $f(\lambda)$ (i.e., the resultant of $f(\lambda)$ and $f'(\lambda)$).

Theorem 2 *Let δ be a positive real number and $\mathcal{A} : a(x, y) = XAX^T = 0$ be a non degenerate conic (not a circle). The implicit equation of $\mathcal{A}_\delta$ agrees with the discriminant (with respect to λ) of*

$$\begin{aligned} f(\lambda) \;=\; & \left(-\delta^2\right)\lambda^3 + \left(a(x, y) - \delta^2(a_{11} + a_{22})\right)\lambda^2 + \\ & + \left(\left(a_{11}a_{22} - a_{12}^2\right)\left(x^2 + y^2 - \delta^2\right) + 2\left(a_{13}a_{22} - a_{12}a_{23}\right)x + 2\left(a_{11}a_{23} - a_{12}a_{13}\right)y + \right. \\ & \left. + a_{11}a_{33} - a_{13}^2 + a_{22}a_{33} - a_{23}^2\right)\lambda + \det(A). \end{aligned}$$

Taking into account that the component of degree 8 in the discriminant (with respect to λ) of $f(\lambda)$ is

$$\left(a_{11}a_{22} - a_{12}^2\right)^2\left(x^2 + y^2\right)^2\left(a_{11}x^2 + 2a_{12}xy + a_{22}y^2\right)^2$$

and the component of degree 7 is

$$-4\left(a_{11}a_{22} - a_{12}^2\right)\left(x^2 + y^2\right)\left(a_{11}x^2 + 2a_{12}xy + a_{22}y^2\right)\left(\left(a_{11}a_{12}a_{23} + \ldots\right)x^3 + \ldots - \left(2a_{11}a_{22}a_{23} + \ldots\right)y^3\right)$$

the proof of Theorem 2 is a direct consequence of the previous arguments together with the fact that the offset of the ellipse (not a circle) and the hyperbola have degree 8 and the offset of the parabola has degree 6 (see [1]). Since a non degenerate conic $\mathcal{A} : XAX^T = 0$ is a parabola if and only if $a_{11}a_{22} - a_{12}^2 = 0$, the degree 8 and 7 components of the discriminant (with respect to λ) of $f(\lambda)$ vanish in this case.

5 On the implicit equation for the δ-offset to a quadric.

The equation of any quadric $\mathcal{A}$ in $\mathbb{R}^3$ can be written as

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz + 2a_{14}x + 2a_{24}y + 2a_{34}z + a_{44} = 0$$

or in matricial form

$$\begin{pmatrix} x & y & z & 1 \end{pmatrix} A \begin{pmatrix} x & y & z & 1 \end{pmatrix}^T = 0$$

where A is the corresponding symmetric matrix. Given two quadrics $\mathcal{A} : XAX^T = 0$ and $\mathcal{B} : XBX^T = 0$, their characteristic equation is defined as $f(\lambda) = \det(\lambda B + A) = \det(B)\lambda^4 + \ldots + \det(A)$ which is, if $\det(B) \neq 0$, a quartic polynomial in λ with real coefficients.

We say that two non degenerate quadrics are touching each other externally if they are separated by a plane tangent to both quadrics. Next theorem characterises when two non degenerate quadrics touch each other externally in terms of the real roots of its characteristic equation (see [5]).

Theorem 3 *Let $\mathcal{A} : XAX^T = 0$ be a non degenerate quadric, $\mathcal{B} : XBX^T = 0$ a sphere and $f(\lambda) = \det(A + B\lambda)$ their characteristic equation. $\mathcal{A}$ and $\mathcal{B}$ touch each other externally if and only if $f(\lambda)$ has a positive double root.*

Next theorem presents a new determinantal description for the implicit equation of $\mathcal{A}_\delta$ when $\mathcal{A}$ is a non degenerate quadric (not a sphere).

Theorem 4 *Let δ be a positive real number and $\mathcal{A} : XAX^T = 0$ be a non degenerate quadric (not a sphere). The implicit equation of $\mathcal{A}_\delta$ agrees with the discriminant (with respect to λ) of the polynomial $f(\lambda) = \det(A + \lambda B)$ where B is the matrix representing the sphere of center (x, y, z) and radius δ.*

6 Conclusion

A new determinantal description for the implicit equation of the offset to non degenerate conics and quadrics has been introduced. The implicit equation of the offset is obtained as the discriminant of a cubic polynomial for conics or as the discriminant of a quartic polynomial for quadrics. In the first case the implicit equation is the determinant of a 5×5 matrix (that can be easily reduced to a 4×4 matrix) whose entries are degree two polynomials in x and y. In the second case the implicit equation is the determinant of a 7×7 matrix (that can be easily reduced to a 6×6 matrix) whose entries are degree two polynomials in x, y and z. These determinantal equations for the offsets of conics and quadrics can be very useful for solving intersection problems involving these geometric constructions. As a final example, by using Theorem 2 and columns and rows manipulation in the matrix providing the λ-discriminant of $f(\lambda)$, the implicit equation of the offset to distance δ for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is equal to

$$E(x,y) = \begin{vmatrix} 1 & 0 & H(x,y) & -2\delta^2a^2b^2 \\ T(x,y) & 2H(x,y) & -3\delta^2a^2b^2 & 0 \\ 0 & T(x,y) & 2H(x,y) & -3\delta^2a^2b^2 \\ 0 & 3 & -2T(x,y) & H(x,y) \end{vmatrix} = 0$$

where

$$T(x,y) = x^2 + y^2 - \delta^2 - a^2 - b^2 \qquad H(x,y) = a^2y^2 + b^2x^2 - \delta^2a^2 - \delta^2b^2 - a^2b^2.$$

The problem of intersecting this offset with a parametric curve $(x(t), y(t))$ is thus reduced to solve the equation $E(x(t), y(t)) = 0$ that, after the linearisation of the corresponding matrix polynomial, can further translated to a generalised eigenvalue problem (not requiring to compute the determinant).

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