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Unboundedness of Betti Numbers of Curves

Ranjana Mehta
Discipline of Mathematics, IIT Gandhinagar, Palaj, Gandhinagar, Gujarat 382355, INDIA.
ranjssj16@gmail.com

Joydip Saha
Discipline of Mathematics, IIT Gandhinagar, Palaj, Gandhinagar, Gujarat 382355, INDIA.
saha.joydip56@gmail.com

Indranath Sengupta
Discipline of Mathematics, IIT Gandhinagar, Palaj, Gandhinagar, Gujarat 382355, INDIA.
indranathsg@iitgn.ac.in

Abstract

Bresinsky defined a class of monomial curves in $\mathbb{A}^4$ with the property that the minimal number of generators or the first Betti number of the defining ideal is unbounded above. We prove that the same behaviour of unboundedness is true for all the Betti numbers and construct an explicit minimal free resolution for this class. We also propose a general construction of such curves in arbitrary embedding dimension.

1 Introduction

Γ is a subset of the set of nonnegative integers $\mathbb{N}$, closed under addition, contains zero and generates $\mathbb{Z}$ as a group. It follows that (see [10]) the set $\mathbb{N} \setminus \Gamma$ is finite and that the semigroup Γ has a unique minimal system of generators $n_0 < n_1 < \cdots < n_p$. The greatest integer not belonging to Γ is called the *Frobenius number* of Γ, denoted by $F(\Gamma)$. The integers n_0 and $p+1$ are known as the *multiplicity* $m(\Gamma)$ and the *embedding dimension* $e(\Gamma)$, of the semigroup Γ. The *Apéry set* of Γ with respect to a non-zero $a \in \Gamma$ is defined to be the set $\mathrm{Ap}(\Gamma, \mathrm{a}) = \{\mathrm{s} \in \Gamma \mid \mathrm{s} - \mathrm{a} \notin \Gamma\}$. A numerical semigroup Γ is *symmetric* if $F(\Gamma)$ is odd and $x \in \mathbb{Z} \setminus \Gamma$ implies $F(\Gamma) - x \in \Gamma$.

This is a study of certain classes of curves from the perspective of boundedness of the Betti numbers of their coordinate rings. The following questions create the focal points for our study and we have managed to answer them partially during the course of this work.

1. Given e (embedding dimension), is it true that for all symmetric numerical semigroups with embedding dimension e, the number of minimal relations is a bounded function of e?

2. Given e (embedding dimension), how does one naturally generalize the examples of Bresinsky and Moh to construct examples of curves in embedding dimension e, having unbounded Betti numbers.

2 Conclusion

Symmetric Numerical Semigroups

The question that given e(embedding dimension), is it true that for all symmetric numerical semigroups with embedding dimension e the number of minimal relations is a bounded function of e. In [6] Herzog proved that the number of minimal relations for a symmetric numerical semigroup of embedding dimension $e = 3$, is 2. It was proved in [2] and [3] respectively that, for $e = 4$ and for certain cases in $e = 5$, the symmetry condition on the semigroup imposes an upper bound on the cardinality of a minimal presentation of a numerical semigroup Γ with embedding dimension e. This remains an open question in general whether symmetry condition on the numerical semigroup $e \geq 5$ imposes an upper bound on the cardinality of a minimal presentation of a numerical semigroup Γ.

Rosales [9] constructed a class of numerical semigroups which are symmetric and showed that the cardinality of a minimal presentation of the semigroup is a bounded function of the embedding dimension e. Our aim is to construct similar classes of numerical semigroups by introducing another parameter d. The value $d = 1$ gives nothing but Rosales' construction and therefore our results generalize results in [9]. What we would like to point out is that the value of $d = 1$ does not play any special role in the construction in [9]. What is perhaps required is a semigroup generated by a sequence of integers $m, m + d, n, n + d, n + 2d, \ldots, n + (e - 3)d$, for suitable positive integers m, n, d. This can be seen as a concatenation of two arithmetic sequences with the same common difference d. Moreover, with suitable condition on these integers we can make sure that this concatenation can not be completed to a complete arithmetic sequence with first term m and common difference d and form a minimal set of generators of the semigroup. We generalize Rosales' construction by defining the symmetric numerical semigroups $\Gamma_{(e,q,d)}(\mathcal{S})$ and $\Gamma_{(e,q,d)}(\mathcal{T})$ of embedding dimension e. We calculate the Apéry sets of $\Gamma_{(e,q,d)}(\mathcal{S})$ and $\Gamma_{(e,q,d)}(\mathcal{T})$. Then we show that $\mu(\Gamma_{(e,q,d)}(\mathcal{S}))$ and $\mu(\Gamma_{(e,q,d)}(\mathcal{T}))$, the cardinality of the set of minimal presentation of $\Gamma_{(e,q,d)}(\mathcal{S})$ and $\Gamma_{(e,q,d)}(\mathcal{T})$ respectively is bounded function of e. The main result is the following:

Theorem 1 1. *Let $e \geq 4$ be an integer, q a positive integer and $m = e + 2q + 1$. Let d be a positive integer that satisfies $\gcd(m, d) = 1$. Let us define $S = \{m, m+d, (q+1)m+(q+2)d, (q+1)m+(q+3)d, \ldots, (q+1)m+(q+e-1)d\}$. The set S is a minimal generating set for the symmetric numerical semigroup $\Gamma_{(e,q,d)}(S)$.*

2. *Let $e \geq 4$ be an integer, q a positive even integer with $q \geq e - 4$ and $m = e + 2q$. Let d be an odd positive integer that satisfies $\gcd(m, d) = 1$. Let us define $T = \{m, m + d, q(m + 1) + (q - \frac{e-4}{2})d + \frac{e}{2}, q(m + 1) + (q - \frac{e-4}{2} + 1)d + \frac{e}{2}, \ldots, q(m + 1) + (q - \frac{e-4}{2} + (e - 3))d + \frac{e}{2}\}$. The set T is a minimal generating set for the symmetric numerical semigroup $\Gamma_{(e,q,d)}(T)$.*

Theorem 2 *The cardinality of a minimal presentation for both the numerical semigroups $\Gamma_{(e,q,d)}(S)$ and $\Gamma_{(e,q,d)}(T)$ is $\frac{e(e-1)}{2} - 1$.*

Monomial Curves and Bresinsky's examples

Let $r \geq 3$ and $n_1, \ldots, n_r$ be positive integers with $\gcd(n_1, \ldots, n_r) = 1$ such that the numbers $n_1, \ldots, n_r$ generate the numerical semigroup $\Gamma(n_1, \ldots, n_r) = \{\sum_{j=1}^{r} z_j n_j \mid z_j \text{ nonnegative integers}\}$ minimally. Let $\eta : k[x_1, \ldots, x_r] \to k[t]$ be the mapping defined by $\eta(x_i) = t^{n_i}, 1 \leq i \leq r$, where k is a field. Let $\mathfrak{p}(n_1, \ldots, n_r) = \ker(\eta)$. Let $\beta_i(\mathfrak{p}(n_1, \ldots, n_r))$ denote the i-th Betti number of the ideal $\mathfrak{p}(n_1, \ldots, n_r)$. Therefore, $\beta_1(\mathfrak{p}(n_1, \ldots, n_r))$ denotes the minimal number of generators $\mathfrak{p}(n_1, \ldots, n_r)$ which is also known as the minimal number of relation for the numerical semigroup $\Gamma(n_1, \ldots, n_r)$. For a given $r \geq 3$, let $\beta_i(r) = sup(\beta_i(\mathfrak{p}(n_1, \ldots, n_r)))$, where is sup is taken over all the sequences of positive integers $n_1, \ldots, n_r$. Bresinsky [1] constructed a class of monomial curves in $\mathbb{A}^4$ to prove that $\beta_1(4) = \infty$. He used this observation to prove that $\beta_1(r) = \infty$, for every $r \geq 4$. Let us recall Bresinsky's example of monomial curves in $\mathbb{A}^4$, as defined in [1]. Let $q_2 \geq 4$ be even. $q_1 = q_2 + 1$, $d_1 = q_2 - 1$. Set $n_1 = q_1q_2$, $n_2 = q_1d_1$, $n_3 = q_1q_2 + d_1$, $n_4 = q_2d_1$.

It is clear that $\gcd(n_1, n_2, n_3, n_4) = 1$. We construct a set of binomials which is a minimal generating set for the ideal $\mathfrak{p}(\underline{\mathbf{n}})$ and also a Gröbner basis with respect to a suitable monomial order. The Gröbner basis helps us in computing the syzygies explicitly and minimally, leading to an explicit minimal free resolution of $\mathfrak{p}(\underline{\mathbf{n}})$. We have proved that $\beta_i(4) = \infty$, for $i = 1, 2, 3$. The main result we prove is the following:

Theorem 3 *Let $S = A_1 \cup A_2' \cup \{g_1, g_2\}$, where $A_2' = \{h_m \mid x_1^m x_4^{(q_1-m)} - x_2^{(q_2-m)} x_3^m, 1 \leq m \leq q_2 - 2\}$.*

1. *S is a minimal generating set for the ideal $\mathfrak{p}(\underline{\mathbf{n}})$;*
2. *S is a Gröbner basis for $\mathfrak{p}(\underline{\mathbf{n}})$ with respect to the lexicographic monomial order induced by $x_3 > x_2 > x_1 > x_4$ on $k[x_1, \ldots, x_4]$;*
3. *$\beta_1(\mathfrak{p}(\underline{\mathbf{n}})) = \mid S \mid = 2q_2$;*
4. *$\beta_2(\mathfrak{p}(\underline{\mathbf{n}})) = 4(q_2 - 1)$;*
5. *$\beta_3(\mathfrak{p}(\underline{\mathbf{n}})) = 2q_2 - 3$.*
6. *A minimal free resolution for the ideal $\mathfrak{p}(\underline{\mathbf{n}})$ over the polynomial ring $R = K[x_1, x_2, x_3, x_4]$ is*
$$0 \longrightarrow R^{2q_2-3} \xrightarrow{P} R^{4(q_2-1)} \xrightarrow{N} R^{2q_2} \longrightarrow R \longrightarrow R/\mathfrak{p}(\underline{\mathbf{n}}) \longrightarrow 0,$$
where
$$P = [\boldsymbol{\delta}_1 \ldots \boldsymbol{\delta}_{q_2-2} \mid \boldsymbol{\xi} \mid \boldsymbol{\zeta} \mid \boldsymbol{\eta} \mid \boldsymbol{\kappa}_1 \ldots \boldsymbol{\kappa}_{q_2-4}]_{4(q_2-1)\times 2q_3-3}$$
$$N = \left[\boldsymbol{\beta}_1 \ldots \boldsymbol{\beta}_{q_2-1} \mid \boldsymbol{\gamma}_1' \ldots \boldsymbol{\gamma}_{q_2-3}' \mid \boldsymbol{\alpha}' \mid \boldsymbol{\beta}_2' \ldots \boldsymbol{\beta}_{q_2-2}' \mid \boldsymbol{\alpha}_1 \ldots \boldsymbol{\alpha}_{q_2-1} \mid -\boldsymbol{\gamma} \mid \boldsymbol{\beta}' \mid \boldsymbol{\gamma}'\right]_{2q_2\times 4(q_2-1)}.$$

The integers n_1, n_2, n_3, n_4 defining Bresinsky's semigroups have the property that $n_1+n_2 = n_3+n_4$. In an attempt to generalize this construction in arbitrary embedding dimension $e \geq 4$ we stumbled upon the family of semigroups $\Gamma_e(M)$, minimally generated by the set $M = \{a, a+d, a+2d, \ldots, a+(e-3)d, b, b+d\}$, where $a = e + 1$, $b > a + (e - 3)d$, $\gcd(a, d) = 1$ and $d \nmid (b - a)$. We first describe the Apéry set $\mathrm{Ap}(\Gamma_e(M), a)$ and used that information to calculate the Frobenius number and a minimal presentation of $\Gamma_e(M)$. However, it turns out that, this does not generalize Bresinsky's class because in this case, for a given e, the number of minimal presentation is indeed bounded above. While $d \nmid (b - a)$ is essential, one also requires a nonlinear polynomial dependence between the integers a and e.

Algebroid Space Curves and Moh's examples

T.T. Moh [7] in 1973 constructed some space curves to study the unboundedness of generators of prime ideals. Let us first recall Moh's curves. Let n be an odd positive integer and $m = (n + 1)/2$. Let λ be an integer such that $\lambda > n(n + 1)m$ and $(\lambda, m) = 1$.
Let $\rho : k[[x, y, z]] \to k[[t]]$ be a mapping defined by
$$\rho(x) = t^{nm} + t^{nm+\lambda},\ \rho(y) = t^{(n+1)m},\ \rho(z) = t^{(n+2)m}.$$
Let $\mathfrak{p}_n = ker\rho$. Moh [8] proved that $\mu(\mathfrak{p}_n) = n + 1$ and therefore for this family of curves in the affine space $\mathbf{A}_k^3$, the minimal number of generators for $\mathfrak{p}_n$ is unbounded above. Moh also produced an explicit generating set for $\mathfrak{p}_n$. However, a minimal free resolution is not known. We have first studied the ideal $P_n = \mathfrak{p}_n \cap k[x, y, z]$. Our objective was to find a Gröbner basis for P_n and see if that throws light on the syzygetic structure of the ideal $\mathfrak{p}_n$ in the power series ring $k[[x, y, z]]$. The study for general n turned out to be hard and therefore restricted our attention to the case $n = 1$. In Chapter 5 we find an explicit set of generators of P_1, which is a Gröbner basis. We finally compute a minimal free resolution of $\mathfrak{p}_1$ over the power series ring $k[[x, y, z]]$. We prove the following theorem:

Theorem 4 *Let us consider the ring* $k[[x,y,z]]$. *Let* $G = x^2 - y - xy^{(3-r)}z^{(r+p-1)} - y^{\frac{1}{2}(r-2)(3r-5)}z^{(-r^2+4r+p-2)}$ *and* $H = xy - [z + y^{(4-r)}z^{(r+p-1)}]$. *Then,*

1. G, H *minimally generate* $\mathfrak{p}_1$;
2. G, H *form a regular sequence;*
3. $\mathfrak{p}_1$ *is minimally resolved by the Koszul complex over the ring* $k[[x,y,z]]$.

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