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Homogenized funtf varieties and algebraic frame completion*

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Abstract

We introduce homogenized funtf (finite tight unit norm frames) varieties and study the degrees of their coordinate projections. These varieties compactify the affine funtf variety differently from the projectivizations studied in [12]. However, each are the closures (Zariski) of the set of finite tight unit norm frames. Our motivation comes from studying the algebraic frame completion problem.

1 Algebraic frame theory

Frames generalize the notion of a basis of a vector space and have found use in numerous fields of science and engineering. For applications, frames of (infinite dimensional) Hilbert spaces are of interest, but in practice only finite frames are used due to the nature of computing. Given a Hilbert space H, a *frame* is a collection of elements $\{f_k\}_{k\in\mathbb{N}} \subset H$ such that there exists A, B with $0 < A \leq B < \infty$ where $A\|h\|^2 \leq \sum_{k\in\mathbb{N}} |\langle h, f_k\rangle|^2 \leq B\|h\|^2$ holds for every $h \in H$. These frame conditions are given by Duffin and Schaeffer in [5]. If $A = B$, then the frame is called *tight*. If H is n-dimensional, then any frame will have $r \geq n$ elements. A frame consisting of elements each of norm one is said to be a *unit norm frame*. Frames which are both tight and unit norm are called *funtf* and are the focus of much research as these frames minimize various measures of error in reconstruction [7, 11]. Algebraic frame theory uses the powerful tools of computational algebraic geometry to solve problems involving finite frame varieties. Such approaches have found success in [3, 6, 12, 14].

Given an $n \times r$ matrix with fixed entries in a few locations, we may ask the question, "What are conditions on the unknown entries such that the columns of this matrix would form a finite unit norm tight frame on $\mathbb{R}^n$?" This is the question of *algebraic frame completion* and is the motivation of our work.

2 Homogenized funtf varieties

Given n and r such that $r \geq n \geq 2$, let W denote an $n \times r$ matrix. The i-th row of W is denoted by v_i, and the j-th column is denoted by w_j. Let $Z_{n,r}$ denote the set of matrices $W \in \mathbb{P}^{r\times n-1}$ such that (1) $v_i v_i^\intercal = v_{i+1}v_{i+1}^\intercal \quad i = 1, \ldots n-1$, (2) $w_j^\intercal w_j = w_{j+1}^\intercal w_{j+1} \quad j = 1, \ldots, r-1$, and (3) the off diagonal entries of WW^T equal zero. Let M_i denote the hypersurface defined by $v_i v_i^\intercal = 0$, and let N_j denote the hypersurface defined by $w_j^\intercal w_j = 0$. We say M_i, (resp. N_j) are the i-th (resp. j-th) *isotropic row (resp. column) hypersurfaces*. Let $Y_{n,r}$ denote the Zariski closure of $Z_{n,r}$ saturated by the

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union of M_i and N_j for $i = 1, \ldots, n$ and $j = 1, \ldots, r$. We say $Y_{n,r}$ is the (n, r) *homogenized funtf variety*, and this is a projective variety. Let $\hat{Y}_{n,r} \subset \mathbb{C}^{n\times r}$ denote the affine cone of $Y_{n,r}$. We define $X_{n,r}$ to be the *affine funtf variety*, whose defining equations may be found in [4]:

$$X_{n,r} = \{W \in \hat{Y}_{n,r} \text{ such that } v_i v_i^\intercal = \frac{r}{n}, i = 1, \ldots, n, \; w_j^\intercal w_j = 1, \; j = 1, \ldots, r\}.$$

In [12], the authors define what they call the *projective funtf variety* which is a variety in $(\mathbb{P}^{n-1})^r$ defined by multihomogeneous functions. This variety is a different compactification than the one we consider.

3 Computing homogenized funtf varieties

For $n = 2$, we compute $Z_{n,r}$ and $Y_{n,r}$ for $r = 3, \ldots, 10$. We find that each component of these varieties has the expected codimension of $n(n-1)/2 + (n-1) + (r-1)$. Moreover we see that there are two components of $Z_{n,r}$ contained in an isotropic hypersurface. These two components each have degree two and are with multiplicity two. We conjecture that the homogenized funtf variety $Y_{n,r}$ has degree $2^{r+n-1} - 8, r > 2, n = 2$, and that this can be proven using a straightforward argument using intersection theory.

Our computational results agree with those reported in Table 1 of [12] in the following sense. For $r = 3, \ldots, 7$ (larger r are not reported) the degrees reported are of $X_{n,r}$ which are twice that of $Y_{n,r}$. We determined the degrees by computing a witness set [13] (the intersection of a variety with a general linear space). For the largest example we computed ($r = 10$ with $\deg Y_{n,r} = 2040$), it took us 213 seconds to compute a witness set using Bertini[1] [2]. Our method was to compute a witness set of $Z_{n,r}$ and then remove the witness points that are in an isotropic hypersurface; this results in a witness set for $Y_{n,r}$. We used [1], a Macaulay2 [8] package to process the files and remove the witness points. If one tries to perform a standard numerical irreducible decomposition, we found that it took an additional 1313 seconds.

4 Degrees of projections

Let X denote the irreducible affine variety that is the restriction of the homogenized funtf variety $Y_{n,r}$ to a general affine chart of $\mathbb{P}^{n\times r-1}$. We are interested in the degree of the fiber of a coordinate projection of X. In particular we are interested in dominant projections where the fiber is finite. To compute the degree of such a fiber, it's sufficient to compute the degree of the fiber over a general point.

We index the coordinate projections of X using 0/1 matrices. Let $\Omega_{n,r,k}$ denote the set of 0/1-matrices of size $n \times r$ with exactly k ones. For ω in $\Omega_{n,r,k}$, let $\pi_\omega \colon X \to \mathbb{C}^k$ denote the projection to the coordinates with a one as an entry in the matrix ω. We denote the degree of the projection $\pi_\omega \colon X \to \mathbb{C}^k$ by $d_\omega(X)$. We say $d_\omega(X)$ is a *coefficient of the multidegree* of X. We call the set $\Omega(X) = \{\omega \in \Omega_{n,r,k} \colon k = \dim X, \pi_\omega \colon X \to \mathbb{C}^k \text{ is dominant}\}$ the *support of* X. The support of X has the following relevance for *algebraic frame completion.* If $\omega \in \Omega(X)$, then a matrix W with general entries where ω has a one can be completed to a point p in $Y_{n,r}$. If p has all real entries, then it is frame up to a constant factor.

Consider $Y_{n,r}$ for $(n, r) = (3, 5)$, which has projective dimension five. Let X denote the restriction of $Y_{n,r}$ to a general affine chart. Then, $\Omega(X)$ is a subset of $\Omega_{3,5,5}$ and contains $\{\omega_1, \ldots, \omega_4\}$, where ω_i and the $d_{\omega_i}(X)$ are as follows:

$$\omega_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \omega_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \omega_3 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \omega_4 = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$d_{\omega_1}(X) = 48 \qquad d_{\omega_2}(X) = 48 \qquad d_{\omega_3}(X) = 96 \qquad d_{\omega_4}(X) = 96.$$

[1]This was done in serial on a Macbook Pro with a 2.6 GHz Intel Core i5 processor and 8 GB 1600 MHz DDR3 memory.

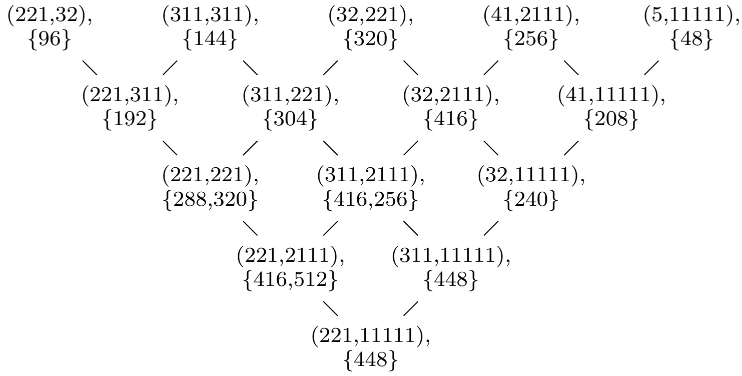

Figure 1

The above hints that many of the $d_\omega(X)$ will be be the same as ω varies. Indeed, this is the case because there is a natural group action of permutation matrices on the left by $\mathfrak{S}_n$ and the right by $\mathfrak{S}_r$, and such an action preserves the multidegree. Thus, we have an equivalence relation on the 0/1 matrices $\Omega_{n,r,k}$ induced by this action. We will use this fact in our computations. Since each equivalence class has a 0/1 matrix with row sums and column sums that are each nonincreasing, then it suffices to consider ω in $\Omega_{n,r,k}$ with this property when determining the $d_\omega(X)$. One would hope that the row sums and column sums of ω would determine the degree, but this is not the case as illustrated next.

We provide pairs of matrices with the same row sums and column sums that yield different values for $d_\omega(X)$. Let $\omega_5,\ldots,\omega_{10}$ be given by the following in order:

$$\begin{bmatrix}1&1&0&0&0\\1&0&1&0&0\\0&0&0&1&0\end{bmatrix}\begin{bmatrix}1&1&0&0&0\\0&0&1&1&0\\1&0&0&0&0\end{bmatrix}\quad\begin{bmatrix}0&1&1&1&0\\1&0&0&0&0\\1&0&0&0&0\end{bmatrix}\begin{bmatrix}1&1&1&0&0\\0&0&0&1&0\\1&0&0&0&0\end{bmatrix}\quad\begin{bmatrix}1&1&0&0&0\\1&1&0&0&0\\0&0&1&0&0\end{bmatrix}\begin{bmatrix}1&1&0&0&0\\1&0&1&0&0\\0&1&0&0&0\end{bmatrix}.$$

We see that for $i=5,7,9$, the row sums and column sums of ω_i and ω_{i+1} are the same. We compute the $d_{\omega_5}(X),\ldots,d_{\omega_{10}}(X)$, to be 512, 416, 256, 416, 288, 320 respectively. We used Bertini[2] to compute the degrees of these fibers using witness sets of projections [10].

We consider the case when $(n,r)=(3,5)$. There are $3003=\binom{15}{5}$ elements in $\Omega(3,5,5)$, and it took 18 hours to compute every $d_\omega(X)$ for $\omega\in\Omega(3,5,5)$. A better approach is to use the structure of the problem as follows. There are only 101 elements in $\Omega(3,5,5)$ that have nonincreasing row sums and column sums, and it takes only 30 minutes to compute these $d_\omega(X)$. These 101 elements may be sorted into 15 sets based on row sums and column sums to better organize the data as seen in Figure 1. This figure displays $d_\omega(X)$ for $(n,r)=(3,5)$ and ω in $\Omega_{3,5,5}$ with nonincreasing row sums and column sums. We arrange the coefficients of the multidegree $d_\omega(X)$ in a lattice according to the product ordering on pairs of partitions representing row sums and column sums. Each node of the lattice consists of (1) a pair of partitions in parentheses and (2) a set of $d_\omega(X)$ where ω is a 0/1 matrix with row and column sums corresponding to that pair of partitions. Note that most of the nodes have a unique element in the set of $d_\omega(X)$; the three exceptions come from $d_{\omega_5},\ldots,d_{\omega_{10}}$ where $\omega_5,\ldots,\omega_{10}$ are given above.

5 Outlook

In this extended abstract we presented preliminary computations regarding homogenized funtf varieties and their coordinate projections. Our results are based on a new homogenization of the affine funtf variety

[2]Note that these computations are performed using homotopy continuation, and it is possible to produce inaccurate results if the software misclassifies points at infinity. To account for this issue, we repeated each computation at least five times.

which led to equations defining a complete intersection consisting of a union of components $Y_{n,r}$ of interest and residual components. Since the degree of $Y_{n,r}$ appears to grow exponentially, we wish for a simpler way to describe the homogenized funtf variety using $d_\omega(X)$. In addition, in the future we would like to study the monodromy group of each of the coordinate projections we considered in the previous section. This monodromy group can capture interesting structure of the algebraic variety as seen in [9] for algebraic varieties coming from kinematics, algebraic statistics, and formation shape control.

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