

Hyperexponential Solutions of Linear Differential Equations over Hyperelliptic Curves

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We consider a linear differential equation

$$a_\ell(x)f^{(\ell)}(x) + \cdots + a_0(x)f(x) = 0, \quad a_i \in \mathbb{K}[x, \sqrt{S(x)}]$$

where $S \in \mathbb{K}[x]$ square free, and $\mathbb{K}$ a finite extension of $\mathbb{Q}$.

Up to homographic transformation, we can assume $\deg S$ is odd.

The differential equation is defined over the curve $\mathcal{C} : y^2 = S(x)$.

Example with $S = x^3 + 3x$, $y = \sqrt{S}$

$$x^2(x-1)^2 f''(x) + (1+y)f(x) = 0$$

Our objective is to present an algorithm to compute the hyperexponential solutions, i.e. solutions f such that

$$f'(x)/f(x) \in \overline{\mathbb{Q}}(x, \sqrt{S(x)}).$$

The algorithm process

- The singularities, values of x for which solutions could be non smooth, are computed.
- Compute the possible local behaviour of the solutions near singular points using suitable base functions.
- A combination of local behaviours give a global candidate.
- When the genus of $y^2 = S(x)$ is $g > 0$, this is not sufficient, $2g$ additional parameters need to be found.

Singularities

Let us note $D(f) = \sqrt{S(x)}\partial_x f$ and rewrite the equation

$$\tilde{a}_\ell(x)D^{(\ell)}(f)(x) + \cdots + \tilde{a}_0(x)f(x) = 0$$

We will use the local parametrization $\phi(\epsilon)$ of $\mathcal{C}$ near $q \in \mathcal{C}$

$$(q_1 + \epsilon, \sqrt{S(q_1 + \epsilon)}) \quad \text{when } q_2 \neq 0$$

$$(q_1 + \epsilon^2, \sqrt{S(q_1 + \epsilon^2)}) \quad \text{when } q_2 = 0$$

Proposition

Let be $q \in \mathcal{C}$, ϕ a local parametrization near q . Assume that at $\epsilon = 0$,

$$\text{val}(\tilde{a}_\ell(\phi(\epsilon))) \leq \min_{i=1 \dots \ell-1} (\text{val}(\tilde{a}_i(\phi(\epsilon)))),$$

then any solution is smooth at q .

Singularities

- 1 Write your equation under the form

$$\tilde{a}_\ell(x)D^{(\ell)}(f) + \cdots + \tilde{a}_0(x)f(x) = 0.$$

- 2 Compute the ideal quotient

$$\mathcal{I} = (y^2 - S(x), \tilde{a}_\ell(x, y)) / (y^2 - S(x), \tilde{a}_\ell(x, y), \dots, \tilde{a}_0(x, y)).$$

- 3 Compute a rational univariate representation of $\mathcal{I}$.
- 4 Using a RootOf notation, return the list of solutions (x, y) up to Galois conjugacy.

Example (continued)

$$x^2(x-1)^2 f''(x) + (1+y)f(x) = 0$$

$\Leftrightarrow$

$$2(x^2+3)(x-1)^2 x D^2(f)(x) - 3y(x^2+1)(x-1)^2 D(f)(x) + 2(x^2+3)^2(1+y)f(x) = 0.$$

At $[\pm i\sqrt{3}, 0]$, the valuations are 2, 1, 4, so $[\pm i\sqrt{3}, 0]$ is singular!

Singularities: $[0, 0]$, $[1, 2]$, $[1, -2]$, $[\text{RootOf}(z^2 + 3), 0]$

Exponential types

Using the local parametrization $\phi(\epsilon)$ near a singular point q , we can write the solutions

$$\exp\left(\int \frac{e(\epsilon)}{\epsilon} d\epsilon\right) (\ln \epsilon)^k (1 + c_1\epsilon + c_2\epsilon^2 + \dots)$$

where

$$e(\epsilon) = \sum_{i=0}^n \frac{r_i}{\epsilon^{i/m}}.$$

The functions e are called generalized exponents.

A generalized exponent is called unramified if $m = 1$, and minimal if there are no other generalized exponents e' such that $e - e' \in \mathbb{N}$.

Definition

An exponential type at $q \in \mathcal{C}$ is an expression of the form

$$E = \frac{P + \sqrt{S}W}{(x - q_1)^m \sqrt{S}}$$

such that there exists an unramified generalized exponent such that

$$\frac{e(\epsilon)}{\epsilon} = \phi'_1(\epsilon)E(\phi(\epsilon)) + O(1).$$

Minimality is then when there are no other exponential type E' and $n \in \mathbb{N}$ such that

$$E - E' = nB_q(x) \text{ with } B_q(x) = \frac{1}{2} \frac{1}{x - q_1} + \frac{1}{2} \frac{q_2}{(x - q_1)\sqrt{S}}.$$

Proposition

Any hyperexponential solution can be written

$$e^{\int \sum_{i=1}^r E_i dx} \left(\prod_{q \in \mathcal{C}} e^{m_q \int B_q(x) dx} \right) e^{\int \frac{P + \sqrt{S}W}{\sqrt{S}} dx}$$

where E_i are exponential types, P, W polynomials, and m_q finitely many non zero positive integers. The pair

$$\left(\sum_{q \in \mathcal{C}} m_q, \frac{P + \sqrt{S}W}{\sqrt{S}} + \sum_{i=1}^r E_i \right)$$

is called the global type of f , and $\sum_{q \in \mathcal{C}} m_q$ is called the exponent difference.

Example (continued): the minimal exponential types found are

$$\text{At } [0, 0]: \left\{ \frac{1+i\sqrt{3}}{2x}, \frac{1-i\sqrt{3}}{2x} \right\}$$

$$\text{At } [\pm i\sqrt{3}, 0]: \{0\}$$

$$\text{At } [1, 2]: \left\{ \frac{(1+i\sqrt{11})(y+2)}{4y(x-1)}, \frac{(1-i\sqrt{11})(y+2)}{4y(x-1)} \right\}$$

$$\text{At } [1, -2]: \left\{ \frac{(1+\sqrt{5})(y-2)}{4y(x-1)}, \frac{(1-\sqrt{5})(y-2)}{4y(x-1)} \right\}$$

There are thus $2 \times 1 \times 2 \times 2 \times 2 = 8$ possible global types.

However, plugging these in the equation gives the following candidates for the exponent difference

$$\pm i\sqrt{3} \pm \frac{i}{2}\sqrt{11} \pm \frac{1}{2}\sqrt{5}$$

None are integers $\Rightarrow$ no hyperexponential solutions!

Hyperexponential solutions

The function $e^{\int B_q(x)dx}$ has a simple zero at q and a simple pole at ∞ , and is smooth everywhere else.

So the divisor of $\prod_{q \in \mathcal{C}} e^{m_q \int B_q(x)dx}$ is

$$\sum_{q \in \mathbb{C}} m_q [q] - n[\infty]$$

$\text{Jac}(\mathcal{C})$ is the quotient space of divisor by principal divisors

Proposition

Any element of $\text{Jac}(\mathcal{C})$ can be uniquely represented by a multi set of $\kappa \leq g$ points $q_1, \dots, q_\kappa \in \mathcal{C}$, non conjugated by the hyperelliptic involution.

Proposition

We can write

$$\prod_{q \in \mathcal{C}} e^{m_q \int B_q(x) dx} = Q e^{\int \frac{\lambda_1 + \dots + \lambda_g x^{g-1}}{\sqrt{S}} dx - \sum_{i=1}^{\kappa} \int B_{q_i}(x) dx}$$

where $q_i \in \mathcal{C}$ are not conjugated, $\kappa \leq g$, and Q a polynomial over $\mathcal{C}$ with $\deg Q \leq g + \sum_{q \in \mathcal{C}} m_q$.

$\Rightarrow$ Once the global type is fixed, we only need to determine the polynomial Q and (at most) $2g$ constants λ, q .

HyperexponentialSols

- ① Compute the minimal exponential types at each singularity.
- ② For every combination of exponential types, insert the candidate in the equation and obtain a global type (n, E) .
- ③ For each global type with $n \in \mathbb{N} \cup \{0\}$, look for solutions

$$Qe^{E + \int \frac{\lambda_1 + \dots + \lambda_g x^{g-1}}{\sqrt{s}} dx - \sum_{i=1}^{\kappa} \int B_{q_i}(x) dx}$$

with Q polynomial over $\mathcal{C}$ of degree $g + n$.

- ④ The space of solutions is a finite union of vector spaces, return a basis of each of them.

Because of the λ, q , the equation to solve is non linear.

The functions $e^{\int \frac{\lambda_1 + \dots + \lambda_g x^{g-1}}{\sqrt{S}} dx - \sum_{i=1}^{\kappa} \int B_{q_i}(x) dx}$ are linearly independent over rational functions for distinct λ, q .

As the solutions form a ℓ dimensional vector space, at most ℓ distinct (λ, q) lead to a non zero solution for Q .

$\Rightarrow$ There are at most ℓ (λ, q) for which there exists a non zero vector space of solutions for Q .

The system can be solved by only solving linear system and zero dimensional algebraic systems, with complexity cost at most $O(n^{(2g+1)\omega})$.

Example

$$10(x^2 - 4x + 3 + (x - 1)\sqrt{x^3 + 3})f''(x) + \\ (7x^4 - 7x^3 + 7x^2 + 27x - 74(7x^2 + 7x + 6)\sqrt{x^3 + 3})f'(x) + \\ 7(x^3 + x + 2 + (x + 2)\sqrt{x^3 + 3})f(x) = 0.$$

Singularities are $[\alpha - 3, -\alpha + 6]$, $[\beta, 0]$, $[1, -2]$, $[1, 2]$
with $\alpha = \text{RootOf}(z^2 - 6z + 15)$, $\beta = \text{RootOf}(z^3 + 3)$.

With respectively exponential types

$$\left\{ 0, \frac{99 - 89\alpha + \sqrt{x^3 + 3}(89 - 29\alpha)}{20\sqrt{x^3 + 3}(x - \alpha + 3)} \right\}, \{0\}, \left\{ \frac{2 - \sqrt{x^3 + 3}}{2\sqrt{x^3 + 3}(x - 1)} \right\}, \left\{ \frac{-4 - 2\sqrt{x^3 + 3}}{\sqrt{x^3 + 3}(x - 1)} \right\}$$

Only one global type is possible

$$\left(3, -\frac{6 + 5\sqrt{x^3 + 3}}{2\sqrt{x^3 + 3}(x - 1)} \right).$$

Algorithm HyperexponentialSols finds one hyperexponential solution

$$e^{-\int \frac{1248x^2 + 3456x - 5760 + (4563x - 5443)\sqrt{x^3 + 3}}{(x-1)(1521x - 1873)\sqrt{x^3 + 3}} dx} (39x^2 - 30x + 119 - 64\sqrt{x^3 + 3})$$

Which simplifies to...

$$\frac{1}{x - 1}$$

Application

Consider for example the Hamiltonian system

$$H = \frac{1}{2}(p_1^2 + p_2^2) + aq_2p_2 + q_2^2p_1 + q_1^3 + cq_2^2q_1^2$$

There is a particular solution for $p_2 = q_2 = 0$ (for energy $H = 1$), which is an elliptic curve.

The variational equation near this curve is

$$f''(x)(2 - 2x^3) - 3x^2f'(x) + (2cx^2 + 2\sqrt{2 - 2x^3} - a^2)f(x) = 0$$

The term $aq_2p_2 + q_2^2p_1$ comes from a magnetic field which breaks the symmetry $p \leftrightarrow -p$ in the Hamiltonian, and makes the $\sqrt{2 - 2x^3}$ unavoidable.

Rewritten with D and $y = \sqrt{2 - 2x^3}$, we have

$$D^2(f)(x) + (2cx^2 - a^2 + by)f(x) = 0$$

$\Rightarrow$ No (finite) singularity!

Two possible global types are found

$$\left(-\frac{i\sqrt{2}}{\sqrt{c}} - 1, -\frac{i\sqrt{2}cx}{\sqrt{2-2x^3}} \right), \left(\frac{i\sqrt{2}}{\sqrt{c}} - 1, \frac{i\sqrt{2}cx}{\sqrt{2-2x^3}} \right)$$

Necessary condition for having a hyperexponential solution $c = -2/(1+n)^2$, $n \in \mathbb{N} \cup \{0\}$.

Exactly one global type is possible $\left(n, -\frac{x}{4\sqrt{2-2x^3}} \right)$

For a given $n \in \mathbb{N} \cap \{0\}$, we look for hyperexponential solutions.

$n = 0$: condition $a = 0$

$$e^{-\int \frac{2x dx}{\sqrt{2-2x^3}}}$$

$n = 1$: condition $a^2 = 2, -1$ give respectively

$$e^{-\int \frac{2x^2 + \sqrt{2-2x^3} + 2x + 2}{(2x+1)\sqrt{2-2x^3}} dx} (2x+1), \quad e^{-\int \frac{2x^2 + \sqrt{2-2x^3} + 2x - 1}{(2x+1)\sqrt{2-2x^3}} dx} (2x+1)$$

$n = 2$: a^2 root of a cubic polynomial, and

$$(17496\sqrt{2-2x^3} + 13122\alpha^2 + (17496x + 136757)\alpha + 78732x + 114258) \times$$

$$e^{-\int \frac{23328\alpha^2 x + 52488\alpha^2 + 174960\alpha x + 46656x^2 + 435074\alpha + 34992\sqrt{2-2x^3} + 157464x + 302925}{\sqrt{2-2x^3}(4\alpha^2 + 36\alpha + 8x + 27)} dx}$$