

An Algorithm for Diagonalizing Matrices of Formal Power Series

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Outline

- ▶ **Background and motivation**

Multiparameter perturbation on formal power series

- ▶ **A general criterion:**

When is a matrix diagonalizable?

- ▶ **Two algorithms:**

When are eigenvalues power series?

How do we complete the diagonalization?

- ▶ **Examples and open problems**

Basic Definitions

Let

$$\mathbb{C}[[x_1, \dots, x_m]]$$

be the ring of **formal power series** with the **involution**

$$c^* = \overline{c}, \quad x_i^* = x_i.$$

For a matrix $A = (a_{ij}) \in \text{Mat}_n(\mathbb{C}[[x_1, \dots, x_m]])$, define its **adjoint** by

$$A^* = (a_{ji}^*).$$

Normal and unitary matrices

$$A \text{ is } \mathbf{normal} \iff AA^* = A^*A,$$

$$U \text{ is } \mathbf{unitary} \iff UU^* = U^*U = I_n.$$

The Diagonalization Problem

Problem

Given a **normal** matrix $A \in \text{Mat}_n(\mathbb{C}[[x_1, \dots, x_m]])$, find

$$U \in U_n(\mathbb{C}[[x_1, \dots, x_m]]), \quad D \in \text{Diag}_n(\mathbb{C}[[x_1, \dots, x_m]])$$

such that

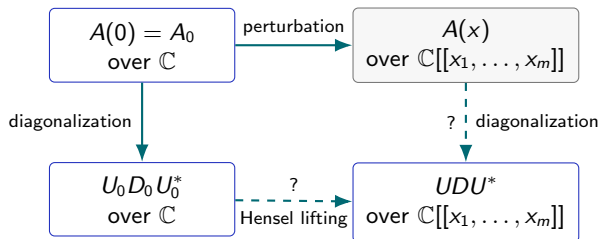
$$A = UDU^*.$$

- ▶ Over $\mathbb{C}$ a normal matrix is always diagonalizable (**spectral theorem**).
- ▶ But $\mathbb{C}[[x_1, \dots, x_m]]$ is not algebraically closed, which leads to some difficulties of detecting diagonalizability.

Perturbation Viewpoint

A matrix over $\mathbb{C}[[x_1, \dots, x_m]]$ can be viewed as a multiparameter perturbation:

$$A(x) = \sum_{\alpha \in \mathbb{N}^m} A_{\alpha} x^{\alpha}, \quad x^{\alpha} = x_1^{\alpha_1} \cdots x_m^{\alpha_m}.$$



Non-degenerate Case and Hensel Lifting

Non-degenerate case

A is non-degenerate if the constant matrix $A(0)$ has **distinct** eigenvalues.

In this case, the spectral decomposition of $A(0)$ can be lifted:

$$A(0) = U_0 D_0 U_0^* \rightsquigarrow A(x) = U D U^*.$$

Hensel lifting determines the lifted data coefficient by coefficient.

Degenerate multiparameter case

The difficulty starts when eigenvalues collide at $x = 0$. Eigenvalues may lie in **algebraic extensions** of power series.

A Previous Sufficient Condition

Parusiński–Rond, 2020

If the discriminant of the minimal polynomial $\mu_A(t)$ is a **normal crossing divisor**, which means

$$\text{disc}(\mu_A) = \text{unit} \cdot \text{monomial},$$

then A is unitarily diagonalizable.

This condition is powerful, but not necessary.

Simple obstruction

The diagonal matrix

$$\begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}$$

is already diagonal, while $\text{disc}(\mu_A) = (x_1 - x_2)^2$ is not normal crossing.

Main Contributions

1. A criterion for diagonalization

For a normal matrix over $\mathbb{C}[[x_1, \dots, x_m]]$:

$$\boxed{\text{unitarily diagonalizable}} \iff \begin{cases} \text{eigenvalues lie in the ring,} \\ \text{spectral projections lie in the ring.} \end{cases}$$

2. An algorithm for diagonalization

For a normal matrix over $\mathcal{O}_{X,o}$:

$$\boxed{\text{eigenvalues in } \hat{\mathcal{O}}_{X,o}} \text{ by Jacobian rank test,}$$
$$\boxed{\text{projections in } \text{Mat}_n(\hat{\mathcal{O}}_{X,o})} \text{ by quotient ideal test.}$$

Criterion I

- ▶ Let $A \in \text{Mat}_n(\mathbb{C}[[x_1, \dots, x_m]])$ be normal.
- ▶ Suppose the distinct eigenvalues are

$$\lambda_1, \dots, \lambda_d.$$

- ▶ The projections satisfy

$$\Pi_k^* = \Pi_k, \quad \sum_k \Pi_k = I_n, \quad \Pi_k \Pi_l = \delta_{kl} \Pi_k, \quad A = \sum_k \lambda_k \Pi_k.$$

Unitary diagonalization criterion (Dai–Liang–Lu–Zhi, 2026)

A is diagonalizable under the adjoint action of $U_n(\mathbb{C}[[x_1, \dots, x_m]])$ **if and only if**

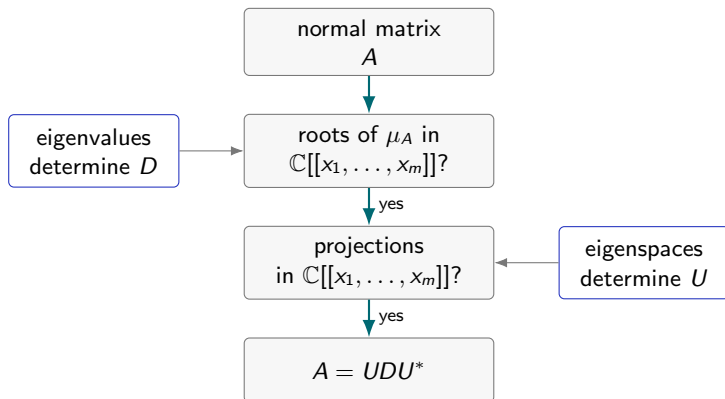
$$\chi_A(t) \text{ splits over } \mathbb{C}[[x_1, \dots, x_m]]$$

and the spectral projections satisfy

$$\Pi_k \in \text{Mat}_n(\mathbb{C}[[x_1, \dots, x_m]]).$$

Criterion I

The diagonalization problem can be divided into two questions.



The two tests control D and U , respectively.

Spectral Projections

Eigenvalues $\rightsquigarrow$ **Lagrange interpolation polynomials**:

$$L_k(t) = \prod_{i \neq k} \frac{t - \lambda_i}{\lambda_k - \lambda_i}, \quad L_k(\lambda_l) = \delta_{kl}.$$

Then the projection is

$$\Pi_k = L_k(A).$$

When the projections are power series

In general, Lagrange interpolation gives entries in the fraction field:

$$p_{ij}^{(k)} \in \text{Frac}(\mathbb{C}[[x_1, \dots, x_m]]).$$

Such an entry belongs to $\mathbb{C}[[x_1, \dots, x_m]]$ iff it can be represented as

$$p_{ij}^{(k)} = \frac{f}{g}, \quad g \in \mathbb{C}[[x_1, \dots, x_m]]^\times.$$

Equivalently, its denominator can be reduced to a **unit**.

Two Small Examples

Diagonalizable

$$A = \begin{pmatrix} \frac{x_1+x_2}{2} & -\frac{i(x_1-x_2)}{2} \\ \frac{i(x_1-x_2)}{2} & \frac{x_1+x_2}{2} \end{pmatrix}$$
$$\mu_A = (t - x_1)(t - x_2)$$

Discriminant not normal crossing, but

$$\Pi_1, \Pi_2 \in \text{Mat}_2(\mathbb{C}[[x_1, \dots, x_m]]).$$

Not diagonalizable

$$A = \begin{pmatrix} x_1^2 & x_1 x_2 \\ x_1 x_2 & x_2^2 \end{pmatrix}$$
$$\mu_A = t(t - x_1^2 - x_2^2)$$

The roots split, but

$$\Pi_k \text{ has denominator } x_1^2 + x_2^2.$$

Algorithmic Setting

- ▶ A general formal power series is not a **finite computer input**.

- ▶ We work with

$$A \in \text{Mat}_n(\mathcal{O}_{X,o}),$$

where X is an affine $\mathbb{C}$ -variety and $o \in X$ is a regular closed point.

- ▶ Its **completion** is a formal power series ring:

$$\hat{\mathcal{O}}_{X,o} \simeq \mathbb{C}[[x_1, \dots, x_m]].$$

Computational target

Decide diagonalizability over $\hat{\mathcal{O}}_{X,o}$ using finite algebraic data from $\mathcal{O}_{X,o}$.

Encoding the Splitting Problem

Let

$$\mu(t) = t^d + c_1 t^{d-1} + \cdots + c_d \in \mathcal{O}_{X,o}[t].$$

Introduce formal roots $\mathbf{y} = (y_1, \dots, y_d)$ by **Vieta's relations**:

$$J = \langle e_k(\mathbf{y}) - (-1)^k c_k \rangle_{k=1}^d \subset \mathcal{O}_{X,o}[\mathbf{y}].$$

Choose

$$Q \in \text{Ass}(\mathcal{O}_{X,o}[\mathbf{y}]/J), \quad L = \kappa(Q).$$

Proposition

The residue field L is a splitting field of $\mu(t)$ over $\text{Frac}(\mathcal{O}_{X,o})$.

Criterion II: A Jacobian Test

Let $\overline{\mathcal{O}_{X,o}}^L$ be the integral closure of $\mathcal{O}_{X,o}$ in L . It is finite over $\mathcal{O}_{X,o}$,

$$\implies \overline{\mathcal{O}_{X,o}}^L = \mathcal{O}_{X,o}[z_1, \dots, z_N]/Q',$$

with $Q' = \langle g_1, \dots, g_D \rangle$, $D \geq N$.

Splitting criterion (Dai–Liang–Lu–Zhi, 2026)

All roots of $\mu(t)$ belong to $\hat{\mathcal{O}}_{X,o}$ if and only if

$$\text{rank} \left(\frac{\partial(g_1, \dots, g_D)}{\partial(z_1, \dots, z_N)} \bmod \mathfrak{n} \right) = N$$

for a maximal ideal $\mathfrak{n}$ over $\mathfrak{m}_{X,o}$.

The full-rank condition means the finite cover is **locally unramified**, so Henselian lifting gives roots in the completion.

Proof Idea: The Rank Test

The proof turns a formal question into a finite geometric one.

- ▶ **Replace completion by Henselization.** For algebraic roots, lying in $\widehat{\mathcal{O}}_{X,o}$ is equivalent to lying in the Henselization $\mathcal{O}_{X,o}^h$.
- ▶ **Translate roots into an étale neighborhood.** The splitting field gives the finite cover $\mathcal{O}_{X,o} \rightarrow \overline{\mathcal{O}_{X,o}}^L$; roots lift exactly when this cover is locally unramified / étale over o .
- ▶ **Detect unramifiedness by differentials.** Writing $\overline{\mathcal{O}_{X,o}}^L = \mathcal{O}_{X,o}[z_1, \dots, z_N]/Q'$, the vanishing of $\Omega_{\overline{\mathcal{O}_{X,o}}^L/\mathcal{O}_{X,o}} \otimes \kappa(\mathfrak{n})$ is the full-rank Jacobian condition.

Rellich's Example: A Failed Jacobian Test

Consider the Hermitian matrix

$$A = \begin{pmatrix} 2x_1 & x_1 + x_2 \\ x_1 + x_2 & 2x_2 \end{pmatrix}.$$

Its minimal polynomial is

$$\mu_A(t) = t^2 - 2(x_1 + x_2)t - (x_1 - x_2)^2.$$

The Vieta ideal is

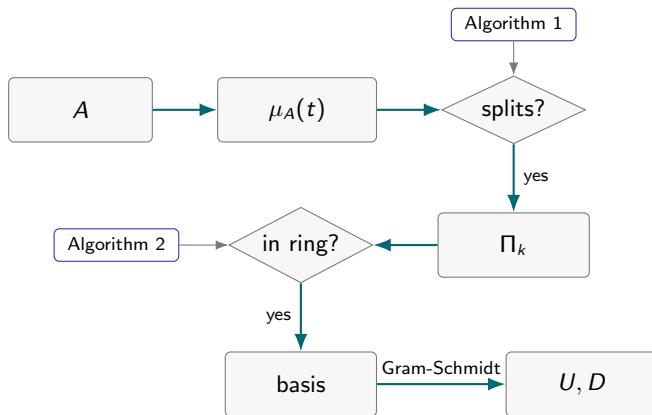
$$J = \langle y_1 + y_2 - 2x_1 - 2x_2, y_1y_2 + (x_1 - x_2)^2 \rangle.$$

Here $Q' = J$. At $\mathfrak{n} = \langle x_1, x_2, y_1, y_2 \rangle$,

$$\left(\frac{\partial g_i}{\partial y_j} \right) = \begin{pmatrix} 1 & 1 \\ y_2 & y_1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \pmod{\mathfrak{n}}.$$

Since the rank is $1 < 2$, $\mu_A(t)$ does not split over $\hat{\mathcal{O}}_{X,o}$.

From A to UDU^*



Failure exits

If either decision node fails, the algorithm returns NOT Diagonalizable.

Algorithm 2: Denominator Test

The crucial step is deciding whether each entry of a spectral projection lies in the completed local ring.

Projection entry

$$R := \mathcal{O}_{X,o}[y, w]/Q',$$

$$\Pi_k = P_k(A),$$

$$p = p_{ij}^{(k)} = \frac{F}{G} \in \text{Frac}(R).$$

The displayed denominator G need not be a unit.

Reduction

Compute the quotient ideal

$$\langle G, Q' \rangle : \langle F, Q' \rangle.$$

If it contains $g \not\equiv 0 \pmod{\mathfrak{n}}$, then

$$p \equiv \frac{f}{g} \pmod{Q'}.$$

Local-ring test

Since $g \notin \mathfrak{n}$, it is a unit in the completed local ring. Hence $p \in \widehat{\mathcal{O}}_{X,o}$. If no such unit denominator exists for some $p_{ij}^{(k)}$, then $\Pi_k \notin \text{Mat}_n(\widehat{\mathcal{O}}_{X,o})$ and the algorithm stops.

Rellich's Example After a Blow-up

The original Rellich matrix fails the Jacobian test. After the **blow-up** $x_1 \mapsto x_1 x_2$, consider

$$\tilde{A} = \begin{pmatrix} 2x_1 x_2 & x_1 x_2 + x_2 \\ x_1 x_2 + x_2 & 2x_2 \end{pmatrix}.$$

Splitting Test

The Jacobian matrix has full rank at $\mathfrak{n}$. Hence

$$\mu_{\tilde{A}}(t) \text{ splits in } \hat{\mathcal{O}}_{X,o}.$$

Spectral Projection Test

Lagrange interpolation gives Π_1, Π_2 . Every entry has denominator

$$4(x_1^2 + 1) \notin \mathfrak{n},$$

so $\Pi_1, \Pi_2 \in \text{Mat}_2(\hat{\mathcal{O}}_{X,o})$.

Conclusion

$\tilde{A}$ is diagonalizable.

Open Problems

1. Diagonalization after finite blow-ups?

Rellich's example is not diagonalizable at the origin, but becomes diagonalizable after a blow-up.

$$A \not\sim UDU^* \implies_{\text{blow-up}} \tilde{A} = UDU^*.$$

Does every normal matrix over a local ring of regular functions become unitarily diagonalizable after finitely many blow-ups?

2. Beyond the current algebraic setting?

Our algorithms use input data from the regular local ring $\mathcal{O}_{X,o}$.

$$A \in \text{Mat}_n(\mathcal{O}_{X,o}) \rightsquigarrow \text{diagonalization over } \hat{\mathcal{O}}_{X,o}.$$

Can the method be extended to matrices over other local rings? Can the projection test be made more efficient?

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