

Learning unions of intersecting affine modules in one dimension with queries

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ISSAC 2026

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Introduction

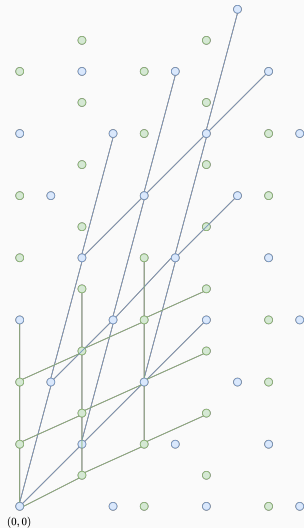
We are interested in learning subsets of $\mathbb{Z}$

Previous works studied the learnability of unions of modules

A union of modules is a set of the form

$$\bigcup_{i=1}^n \sum_{j=1}^m b_{i,j} \mathbb{Z}, \quad b_{i,j} \in \mathbb{N}^d$$

For learnability, it is important whether $b_{i,j}$ are given in unary or binary notation



Unary representation

- Finite unions of modules are polynomially learnable - Takada, Abe

Binary representation

- Single modules are polynomially learnable - Helmbold, Sloan, Warmuth
- In the PAC model, learning unions of modules is as hard as learning DNF - Abe

Our contribution: finite unions of one-dimensional intersecting affine modules are exactly learnable in polynomial time in binary representation using queries

$$\mathcal{C}(\cup \text{ affine modules}) \supseteq \mathcal{C}(\cup \text{ intersecting affine modules}) \supseteq \mathcal{C}(\cup \text{ modules})$$

The class of sets that we learn

One-dimensional affine module: $[a]_b := a + b\mathbb{Z}$ where $a, b \in \mathbb{N}$

Union of intersecting affine modules: $U_* = \bigcup_{i=1}^k [a_i]_{b_i}$ where $\bigcap_{i=1}^k [a_i]_{b_i} \neq \emptyset$

$[0]_5, [0]_3, [0]_2$



$[0]_5 \cup [0]_3 \cup [0]_2$



Learning framework

The learner interacts with a teacher who knows the target set U_*



Queries

- Equivalence (**EQ**): $H = U_*$?
 - YES $\rightarrow$ learning succeed
 - NO $\rightarrow$ returns a counterexample
 $x \in U_* \setminus H$
- Membership (**MQ**): $x \in U_*$?
- Subset (**SQ**): $M \subseteq U_*$?

Goal: reconstruct U_* exactly

Main result

Theorem

The class of unions of **intersecting** affine modules is efficiently exactly learnable with $=$ and $\subseteq$ when the coefficients are given in **binary**.

Complexity parameters:

- $size(\text{largest counterexample})$
- $\# \text{ modules smallest representation} \leq size(\text{smallest representation})$

The learner starts with no information about the structure

Target set U_* (unknown)



Initial hypothesis: $H = \emptyset$

Learning process

A module is created with the counterexample as representative and period 0



EQ: $\emptyset = U_*$? NO
Counterexample: 0

$$H : \emptyset \rightsquigarrow [0]_0$$

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After the second counterexample the period is refined using the gcd



EQ: $[0]_0 = U_*$? NO

Counterexample: 15

Period refinement: $15 = \gcd(0, 15 - 0)$

SQ: $[0]_{15} \subseteq U_*$? YES

$H : [0]_0 \rightsquigarrow [0]_{15}$



Subset queries validate a refinement of the module



EQ: $[0]_{15} = U_*$? NO

Counterexample: 22

$$1 = \gcd(15, 22 - 0)$$

SQ: $[0]_1 \subseteq U_*$? **NO**

$$H_{\times} : [0]_1$$



If the subset query fails, we keep the old hypothesis module



EQ: $[0]_{15} = U_*$? NO

Counterexample: 22

$H : [0]_{15}$

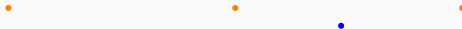
If no module in the hypothesis is refined, a new one is created



EQ: $[0]_{15} = U_*$? NO

Counterexample: 22

$$H : [0]_{15} \rightsquigarrow [0]_{15} \cup [22]_0$$



Maximal modules and coverings



Maximal module M:

- if $\nexists M'$ such that $M \subsetneq M' \subseteq U^*$
- M cannot be extended inside U_*

- Maximal modules of our target



Maximal covering of M: set of maximal modules containing M

- Maximal covering of orange



- Maximal covering of blue



Property 1: The smallest representation of U_* is given by its maximal modules

Property 2: hypothesis modules have pairwise disjoint maximal coverings

Intuition for correctness and efficiency

A module is refined by counterexamples contained in its maximal covering



EQ: $[0]_{15} \cup [22]_0 = U_*$? NO

Counterexample: 27

$H : [0]_{15} \cup [22]_0$



Intuition for correctness and efficiency

A module is refined by counterexamples contained in its maximal covering



EQ: $[0]_{15} \cup [22]_0 = U_*$? NO

Counterexample: 27

$$3 = \gcd(15, 27 - 0)$$

SQ: $[0]_3 \subseteq U_*$? YES

$$H : [0]_{15} \cup [22]_0$$



Intuition for correctness and efficiency

The new maximal covering is contained in the old maximal covering



EQ: $[0]_{15} \cup [22]_0 = U_*$? NO

Counterexample: 27

$$3 = \gcd(15, 27 - 0)$$

SQ: $[0]_3 \subseteq U_*$? YES

$$H : [0]_{15} \cup [22]_0 \rightsquigarrow [0]_3 \cup [22]_0$$



Intuition for correctness and efficiency

A module is refined by counterexamples contained in its maximal covering



EQ: $[0]_{15} \cup [22]_0 = U_*$? NO

Counterexample: 27

$$5 = \gcd(0, 27 - 22)$$

SQ: $[22]_5 \subseteq U_*$? NO



Intuition for correctness and efficiency

A module is refined by counterexamples contained in its maximal covering



EQ: $[0]_{15} \cup [22]_0 = U_*$? NO

Counterexample: 27

$$5 = \gcd(0, 27 - 22)$$

SQ: $[22]_5 \subseteq U_*$? NO

$$H : [0]_3 \cup [22]_0$$



Intuition for correctness and efficiency

Modules become maximal after polynomially many refinements

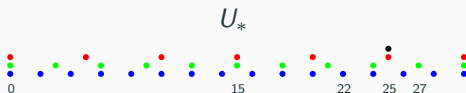


$$H : [0]_3 \cup [22]_0$$



Intuition for correctness and efficiency

No further module is created, since $\#$ hypothesis modules $\leq \#$ maximal modules



EQ: $[0]_3 \cup [22]_0 = U_*$? NO

Counterexample: 25

$$H : [0]_3 \cup [22]_0 \rightsquigarrow [0]_3 \cup [22]_0 \cup [25]_0$$



Intuition for correctness and efficiency

Each counterexample refines at least one module



EQ: $[0]_3 \cup [22]_0 \cup [25]_0 = U_*$? NO

Counterexample: 10

$12 = \gcd(0, 22 - 10)$; $15 = \gcd(0, 25 - 10)$

SQ: $[22]_{12}, [25]_{15} \subseteq U_*$? YES

$H : [0]_3 \cup [22]_{12} \cup [25]_{15}$



Intuition for correctness and efficiency

Each counterexample refines at least one module



EQ: $[0]_3 \cup [22]_{12} \cup [25]_{15} = U_*$? NO

Counterexample: 8

$$2 = \gcd(12, 22 - 8)$$

SQ: $[22]_2 \subseteq U_*$? YES

$H : [0]_3 \cup [22]_2 \cup [25]_{15}$



Intuition for correctness and efficiency

Each hypothesis module converges to a distinct maximal module



EQ: $[0]_3 \cup [22]_2 \cup [25]_{15} = U_*$? NO

Counterexample: 5

$$5 = \gcd(15, 25 - 5)$$

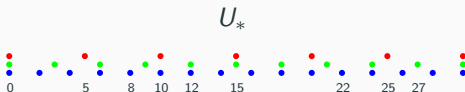
SQ: $[25]_5 \subseteq U_*$? YES

$H : [0]_3 \cup [22]_2 \cup [25]_5$



Intuition for correctness and efficiency

The hypothesis is now equal to the target



EQ: $[0]_3 \cup [22]_2 \cup [25]_5 = U_*$? YES



Summary

- Properties of unions of one-dimensional intersecting affine modules
- Notions for establishing the correctness of the learning algorithm
- A new efficiently exactly learnable class in the query model

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