



POLITÉCNICA



Gröbner Bases of Burchnall-Chaundy Ideals for Ordinary Differential Operators

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Outline

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- 1 Ordinary Differential Operators and centralizers

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Gröbner Bases

- ① Ordinary Differential Operators and centralizers
- ② Goodearl's Theorem
- ③ Burchnall-Chaundy Ideals

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- 1 Ordinary Differential Operators and centralizers
- 2 Goodearl's Theorem
- 3 Burchnall-Chaundy Ideals
- 4 Gröbner basis for BC-ideal
- 5 Applied results

Ordinary differential operators

Goal of the talk

Our running example

$$\mathbf{L} = \partial^4 - 24\wp(x)\partial^2 - 36\wp'(x)\partial - (72\wp(x)^2 + 4)$$

$\wp'(x)^2 = 4\wp(x)^3 - \wp(x) + \frac{1}{216}$ is a $\wp$ -Weierstrass elliptic function.

The Baker-Akhiezer function

A special **eigenfunction** $\Psi(x)$ of $\mathbf{L}$.

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Theorem [Krichever (1977)]

If $\mathbf{L}$ and $\mathbf{B}$ commute, there is a common eigenfunction Ψ such that

$$\mathbf{L}\Psi(x) = \lambda\Psi(x), \quad \mathbf{B}\Psi(x) = \mu\Psi(x).$$

Goal: compute Ψ

Ordinary Differential Operators

- (K, ∂) computable.
- $\mathcal{C}$ field of constants known.

Ring of linear differential operators

- $K[\partial]$ non-commutative polynomials in ∂ .
- $\partial a = a\partial + \partial(a)$ for $a \in K$.

Ordinary Differential Operators

$$L = \partial^n + u_2 \partial^{n-2} + \cdots + u_n$$

Centralizer

$$\mathcal{Z}(\mathbf{L}) = \{B \in K[\partial] : \mathbf{L}B - B\mathbf{L} = [\mathbf{L}, B] = 0.\}.$$

Basic properties

- ① $\mathcal{C}[\mathbf{L}] \subset \mathcal{Z}(\mathbf{L})$.
- ② $\mathcal{Z}(\mathbf{L})$ is a $\mathcal{C}[\mathbf{L}]$ -module.
- ③ $\mathcal{Z}(\mathbf{L})$ is a $\mathcal{C}$ -algebra.

Goodearl's Theorem

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Theorem [Goodearl (1983)]

Let $\mathcal{O}$ be the set of orders in $\mathcal{Z}(L) \pmod{n}$.

- $\mathcal{O}$ is an additive subgroup of $\mathbb{Z}_n$.
- $\{G_o : o \in \mathcal{O}\}$ generates $\mathcal{Z}(L)$ (as a $\mathcal{C}[L]$ -module).
- G_o is monic, has order $d_o \equiv o \pmod{n}$, minimal with this property.

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$$\mathcal{Z}(L) = \bigoplus_{o \in \mathcal{O}} G_o \mathcal{C}[L]$$

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[JP,Rueda (2025)]

Given L , there is an algorithm to compute $\{G_o : o \in \mathcal{O}\}$.

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$$\begin{aligned} G_2 = & -(75600\wp^4 + 15120\wp^2 - 50\wp - 189)\partial^2 \\ & + (-37800\wp^2 - 1890)\wp'\partial^3 - (37800\wp^3 + 5670\wp - 25)\partial^4 \\ & - (6300\wp\wp')\partial^5 - (3060\wp^2 + 366)\partial^6 - (270\wp')\partial^7 \\ & - (60\wp)\partial^8 + \partial^{10} \end{aligned}$$

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$$\begin{aligned} G_3 = & \partial - (504\wp\wp')\partial^2 - (756\wp^2 + 105)\partial^3 - (126\wp')\partial^4 \\ & - (42\wp)\partial^5 + \partial^7 \end{aligned}$$

The Burchnall-Chaundy ideal for a linear differential operator

Burchall-Chaundy polynomials

Theorem [Burchall-Chaundy (1923)]

If L and B commute, then there is a non-zero polynomial $f(\lambda, \mu) \in \mathcal{C}[\lambda, \mu]$ such that

$$f(L, B) = 0.$$

Burchall-Chaundy polynomials

Generalization for several operators

Let $L, B_1, \dots, B_k$ be commuting operators. We say that $f(\lambda, \mu_1, \dots, \mu_k) \in \mathcal{C}[\lambda, \mu_1, \dots, \mu_k]$ is a **Burchall-Chaundy polynomial** if

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Burchall-Chaundy ideal

The **Burchall-Chaundy ideal** of L ($\text{BC}(L)$) is the ideal of Burchall-Chaundy polynomials for L together with its Goodearl's basis $\{G_o : o \in \mathcal{O}\}$.

$$\mathcal{Z}(L) = \bigoplus_{o \in \mathcal{O}} G_o \mathcal{C}[L] \longleftrightarrow \mathcal{Z}(L) \simeq \frac{\mathcal{C}[\lambda, \mu_o : o \in \mathcal{O}]}{\text{BC}(L)}$$

Representation in Goodearl's basis

To compute $\text{BC}(L)$, we will solve the following problem:

Representation problem

Given an operator $P \in K[\partial]$, find polynomials $p_o(\lambda)$ for $o \in \mathcal{O}$ such that

$$P = \sum_{o \in \mathcal{O}} p_o(L) G_o + R,$$

such that $P \in \mathcal{Z}(L)$ if and only if $R = 0$.

Representation in Goodearl's basis

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Representation problem

$$P = \sum_{o \in \mathcal{O}} p_o(L) G_o + R,$$

Division-like step

- Let $p = \text{ord}(P)$, find $o_p \equiv p \pmod{n}$.
- Write $p = qn + \text{ord}(G_{o_p})$.
- Compute $\tilde{P} = P - c_p L^q G_{o_p}$ has lower order than P .
- Repeat until something goes wrong.

Gröbner Basis of the Burchnall-Chaundy Ideal for a Ordinary Differential Operator

Degree two elements

Trivial remark

If $P, Q \in \mathcal{Z}(\textcolor{teal}{L})$, then $PQ \in \mathcal{Z}(\textcolor{teal}{L})$.

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If $G_{o_1}, G_{o_2} \in \mathcal{Z}(L)$, then $G_{o_1} G_{o_2} \in \mathcal{Z}(L)$.

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If $G_{o_1}, G_{o_2} \in \mathcal{Z}(L)$, then $G_{o_1} G_{o_2} \in \mathcal{Z}(L)$.

The representation of $G_{o_1} G_{o_2}$ in Goodearl's basis gives a polynomial in the BC-ideal:

$$R_{o_1, o_2} = \mu_{o_1} \mu_{o_2} - \sum_{o \in \mathcal{O}} p_o(\lambda) \mu_o.$$

Gröbner basis of the BC-ideal

Degree two elements

$$D_2 = \{R_{o_1, o_2} : o_1, o_2 \in \mathcal{O} \setminus \{0\}\}.$$

Generation

D_2 is a basis of $\text{BC}(\mathcal{L})$.

Gröbner basis of the BC-ideal

Degree two elements

$$D_2 = \{R_{o_1, o_2} : o_1, o_2 \in \mathcal{O} \setminus \{0\}\}.$$

Generation

D_2 is a basis of $\text{BC}(\mathcal{L})$.

Gröbner basis

D_2 is a **Gröbner basis** of $\text{BC}(\mathcal{L})$.

Example continued

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$$G_0 \rightarrow 0, \quad G_1 \rightarrow 5, \quad G_2 \rightarrow 10, \quad G_3 \rightarrow 7.$$

$$D_2 = \{\mu_1^2 - \mu_2\}$$

$$G_1 G_1 \rightarrow 10 \longrightarrow G_2$$

Example continued

$$\mathbf{L} = \partial^4 - 24\wp(x)\partial^2 - 36\wp'(x)\partial - (72\wp(x)^2 + 4)$$

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$$D_2 = \{\mu_1^2 - \mu_2, \mu_1\mu_2 - (\lambda^2 - 25\lambda + 154)\mu_3 + \left(\frac{\lambda}{2} - 7\right)\mu_1\}$$

$$G_1 G_2 \rightarrow 15 \equiv 3 \pmod{4} \longrightarrow 15 = 2 \cdot 4 + 7$$

Example continued

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$$G_1 G_3 \rightarrow 12 \equiv 0 \pmod{4} \longrightarrow 12 = 3 \cdot 4 + 0$$

Example continued

$$L = \partial^4 - 24\wp(x)\partial^2 - 36\wp'(x)\partial - (72\wp(x)^2 + 4)$$

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$$G_2 G_2 \rightarrow 20 \equiv 0 \pmod{4} \longrightarrow 20 = 5 \cdot 4 + 0$$

Example continued

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$$G_2G_3 \rightarrow 17 \equiv 1 \pmod{4} \longrightarrow 17 = 3 \cdot 4 + 5$$

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$$G_3 G_3 \rightarrow 14 \equiv 2 \pmod{4} \longrightarrow 14 = 3 \cdot 4 + 2$$

Example continued

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Finding the Baker-Akhiezer function in our
example

Recall Krichever's Theorem

Theorem [Krichever (1977)]

If L and B commute, there is a common eigenfunction Ψ such that

$$L\Psi(x) = \lambda\Psi(x), \quad B\Psi(x) = \mu\Psi(x).$$

Hence, if we pick an operator $B \in \mathcal{Z}(L)$ of coprime order with L , Ψ is the solution of the common right factor of $L - \lambda$ and $B - p_B(\lambda, \bar{\mu})$.

The common right factor can be computed using differential subresultants.

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In our example

$$\begin{aligned} \text{grcd}(L - \lambda, G_1 - \mu_1) = & (((72\lambda^2 - 1800\lambda + 11088)\wp + 6\mu_3)\wp' \\ & + 1728\mu_1\wp^4 + (72\lambda\mu_1 - 576\mu_1)\wp^2 \\ & - 2\mu_1\wp - \lambda^2\mu_1 + 25\lambda\mu_1 - 154\mu_1) + \\ & ((-216\mu_1\wp^2 - 6\lambda\mu_1 + 66\mu_1)\wp' \\ & + (-36\lambda^2 + 900\lambda - 5544)\wp^2 - 6\mu_3\wp \\ & + \lambda^3 - 36\lambda^2 + 429\lambda - 1694)\partial \end{aligned}$$

Conclusions

Summary

- Simple Gröbner basis for the BC-ideal.
- Prepared for further results in the spectral differential Galois theory.
- Algorithms implemented in SageMath.

Future work

- What happens with higher rank operators?
- Goodearl's basis algorithm may not terminate.

Thank you!

Contact webpage:

- <https://www.github.com/Antonio-JP/dalgebra>