

Bivariate Differential Dimension Polynomials via Bigraded Rees Construction

Alexander Levin

The Catholic University of America
Washington, D. C. 20064

51st International Symposium on Symbolic and Algebraic
Computation

Oldenburg, July 14, 2026

Let K be a differential field, $\text{Char } K = 0$, with basic set of mutually commuting derivations $\Delta = \{\partial_1, \dots, \partial_m\}$.

We will often use the prefix Δ - as a shorthand for "differential".

Let Θ be the free commutative semigroup generated by Δ .

If $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_m^{\alpha_m} \in \Theta$ ($\alpha_i \in \mathbb{N}$), the **order** of ∂^α is defined as

$$\text{ord } \partial^\alpha = |\alpha| = \sum_{i=1}^m \alpha_i. \quad \text{If } r, s \in \mathbb{N} \text{ and } s \leq r, \text{ we set}$$

$$\Theta(r) = \{\theta \in \Theta \mid \text{ord } \theta \leq r\} \text{ and } \Theta(r, s) = \{\theta \in \Theta \mid s \leq \text{ord } \tau \leq r\}.$$

An expression $f = \sum_{\theta \in \Theta} a_\theta \theta$, where $a_\theta \in K$ and $a_\theta \neq 0$ for almost all $\theta \in \Theta$, is called a **Δ -operator** over K . Its order is defined as

$$\text{ord}(f) = \max\{\text{ord } \theta \mid a_\theta \neq 0\}.$$

The set $\mathfrak{D}$ of all Δ -operators is a ring with the natural structure of a K -vector space and multiplication based on the rule

$$\partial a = a\partial + \partial(a) \quad (a \in K, \partial \in \Delta).$$

If $r, s \in \mathbb{N}$ and $s \leq r$, then $\mathfrak{D}_r$ and $\mathfrak{D}_{r,s}$ denote the vector K -subspaces of $\mathfrak{D}$ generated by $\Theta(r)$ and $\Theta(r, s)$, respectively.

Clearly, $(\mathfrak{D}_r)_{r \in \mathbb{N}}$ is an ascending filtration on $\mathfrak{D}$ and

$$\mathfrak{D}_r \mathfrak{D}_{r'} = \mathfrak{D}_{r+r'}, \quad \mathfrak{D}_{r,s} \mathfrak{D}_{r',s'} = \mathfrak{D}_{r+r',s+s'}.$$

A left $\mathfrak{D}$ -module is called a **Δ - K -module**. Thus, a K -vector space M is a Δ - K -module if elements of Δ act on M as mutually commuting endomorphisms of its additive group and $\partial(ax) = \partial(a)x + a\partial(x)$ for $a \in K$, $x \in M$, $\partial \in \Delta$.

M is finitely generated (or free) if it is finitely generated (respectively, free) as a left $\mathfrak{D}$ -module.

If M and N are two Δ - K -modules, then $\phi : M \rightarrow N$ is said to be a Δ -homomorphism if $\phi(\partial x) = \partial \phi(x)$ for any $x \in M$, $\partial \in \Delta$.

If M is a Δ - K -module, then the maximal number of elements $e_1, \dots, e_k \in M$ such that the set $\{\theta e_i \mid \theta \in \Theta, 1 \leq i \leq k\}$ is linearly independent over K (we say that $e_1, \dots, e_k$ are Δ -linearly independent over K) is called the **Δ -dimension** of M over K ; it is denoted by $\Delta\text{-dim}_K M$.

The following theorem introduces a Hilbert-type dimension polynomial of a finitely generated Δ - K -module.

beamer-tu-1

beamer-ur-log

Theorem 1 (J. Johnson, 1969)

Let $M = \sum_{i=1}^n \mathfrak{D} x_i$ be a finitely generated Δ - K -module. Then there exists a polynomial $\phi(t) \in \mathbb{Q}[t]$ such that

(i) $\phi(r) = \dim_K \sum_{i=1}^n \mathfrak{D}_r x_i$ for all sufficiently large $r \in \mathbb{N}$ (that is, there exists $r_0 \in \mathbb{N}$ such that the equality holds for all $r \geq r_0$).

(ii) $\phi(t) = \sum_{i=0}^m c_i \binom{t+i}{i}$ where $c_i \in \mathbb{Z}$.

(iii) $d = \deg \phi(t)$, c_m and c_d (if $d < m$) do not depend on the generators of M over $\mathfrak{D}$ and $c_m = \Delta\text{-dim}_K M$.

(iv) If $x_1, \dots, x_n$ are Δ -linearly independent, $\phi(t) = n \binom{t+m}{m}$.

The polynomial $\phi(t)$ is called the **differential (or Δ -) dimension polynomial** of M associated with the Δ -generators $x_1, \dots, x_n$.

Methods and algorithms for the computation of differential dimension polynomials can be found in

[Kondratieva, M. V.; Levin, A. B.; Mikhalev, A. V.; Pankratiev, E. V. *Differential and Difference Dimension Polynomials*. Kluwer Acad. Publ., 1998].

In what follows, we use a Rees-type bigraded construction and the Gröbner basis method to prove the following generalization Johnson's theorem.

beamer-tu-1

beamer-ur-log

Theorem 2

Let K be a Δ -field and $M = \sum_{i=1}^n \mathfrak{D} x_i$.

For any $r, s \in \mathbb{N}$, $s \leq r$, let $M_{r,s} = \sum_{i=1}^n \mathfrak{D}_{r,s} x_i$.

Then there exists a polynomial $P_M(r, s) \in \mathbb{Q}[r, s]$ and integers $s_0, d_0 \geq 0$ such that

- (i) $P_M(r, s) = \dim_K M_{r,s}$ for all integers $r \geq s \geq s_0$, $r - s \geq d_0$.
- (ii) Let $P_M^{\text{top}}(r, s)$ denote the homogeneous part of $P_M(r, s)$ of highest total degree. Then $P_M^{\text{top}}(r, s)$ depends only on M , not on the chosen finite generating set.

Equivalently, the coefficients of all monomials $r^i s^j$ with $i + j = \deg P_M$ are generator-independent invariants.

The polynomial $P_M(r, s)$ is called the **bivariate Δ -dimension polynomial** of the Δ - K -module M associated with the system of Δ -generators $\{x_1, \dots, x_n\}$.

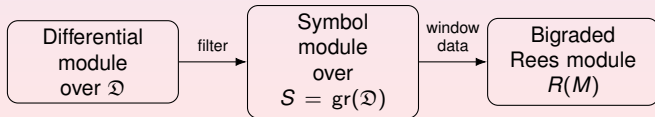
The main idea for the proof of Theorem 2 and for the computation of $P_M(r, s)$ is to work with the graded ring associated with the filtration $(\mathfrak{D}_r)_{r \in \mathbb{N}}$:

$$S := \text{gr}(\mathfrak{D}) \cong K[\xi_1, \dots, \xi_m],$$

where $\xi_i = \partial_i + \mathfrak{D}_0$ is the **symbol** of ∂_i .

At symbol level

Multiplication is homogeneous and order shifts are additive. This makes it possible to encode order windows by a commutative bigraded construction.



Let $M = \sum_{i=1}^n \mathfrak{D}x_i$ and $M_{r,s} = \sum_{i=1}^n \mathfrak{D}_{r,s}x_i$ for $r, s \in \mathbb{N}$, $s \leq r$.

Fix a presentation

$$M \cong F/N, \quad F = \mathfrak{D}e_1 \oplus \cdots \oplus \mathfrak{D}e_n.$$

Define the free order window

$$F_{r,s} = \text{Span}_K\{\partial^\alpha e_i \mid s \leq |\alpha| \leq r, 1 \leq i \leq n\}.$$

Then define

$$Q_{r,s} = F_{r,s}/(F_{r,s} \cap N).$$

The natural map $F \rightarrow M$ induces an isomorphism

$$Q_{r,s} \cong M_{r,s}.$$

Thus, $\dim_K Q_{r,s} = \dim_K M_{r,s}$.

The bigraded Rees-type module

Let

$$A = K[\xi_1, \dots, \xi_m, z, t],$$

with the $\mathbb{N}^2$ -grading

$$\deg(\xi_i) = \deg(z) = (1, 0), \quad \deg(t) = (1, 1).$$

Then

$$z^{r-s}t^s \text{ has bidegree } (r, s).$$

Define

$$R(M) = \bigoplus_{r \geq s \geq 0} Q_{r,s} z^{r-s} t^s.$$

Then the bihomogeneous component satisfies

$$R(M)_{(r,s)} \cong Q_{r,s} \cong M_{r,s}.$$

Hence

$$\dim_K R(M)_{(r,s)} = \dim_K M_{r,s}.$$

The idea of the proof of Theorem 2 is as follows.

After passing to the associated graded data, $R(M)$ is a finitely generated bigraded module over the Noetherian ring A .

Therefore its bigraded Hilbert function is eventually polynomial.

The fact that the homogeneous part $P_M^{\text{top}}(r, s)$ of $P_M(r, s)$ of highest total degree does not depend on the set of generators of M over $\mathcal{D}$ is justified by the following argument.

Let $(M_{r,s})$ and $(M'_{r,s})$ be two bifiltrations of M of the type considered in Theorem 2, which are associated with different systems of generators of M over $\mathcal{D}$. Then there exist $r_0, s_0 \in \mathbb{N}$ such that

$$M_{r-r_0, s-s_0} \subseteq M'_{r,s} \subseteq M_{r+r_0, s+s_0}$$

for all sufficiently large r, s with $s \leq r$.

Hence, the corresponding bivariate Δ -dimension polynomials have equal homogeneous parts of highest total degree.

Thus, the total-degree leading part $P_M^{\text{top}}(r, s)$ of the bivariate Δ -dimension polynomial $P_M(r, s)$ is independent of the chosen generating set.

In other words, the coefficients of the monomials $r^i s^j$ with $i + j = \deg P_M$ are intrinsic invariants of M .

Next we will consider the computation of the bivariate Δ -dimension polynomial.

beamer-tu-1

beamer-ur-log

Filtered-graded reduction

Let

$$\mathrm{gr}(F) \cong S^n, \quad S = K[\xi_1, \dots, \xi_m].$$

For $N \subset F$, use the induced filtration and form

$$\mathrm{gr}(N) \subset \mathrm{gr}(F).$$

For sufficiently large r, s ,

$$\dim_K M_{r,s} = \dim_K \mathrm{gr}(F)_{r,s} - \dim_K \mathrm{gr}(N)_{r,s}.$$

Computational advantage

The right-hand side is a calculation in a commutative graded module over the polynomial ring S .

Computing $\text{gr}(N)$

Suppose

$$N = \langle g_1, \dots, g_t \rangle \subset F.$$

The top-order symbols $\text{in}_{\text{ord}}(g_i)$ do not necessarily generate $\text{gr}(N)$.

Filtered Gröbner step

Compute a filtered Gröbner basis

$$G = \{h_1, \dots, h_q\}$$

of N . Then

$$\text{gr}(N) = \langle \text{in}_{\text{ord}}(h_1), \dots, \text{in}_{\text{ord}}(h_q) \rangle_S.$$

Equivalent implementation

Homogenize with respect to order, compute a homogeneous Gröbner basis, and then pass to top-order symbols.

Algorithm

- 1 Input a presentation

$$M \cong F/N, \quad F = \mathfrak{D}^n, \quad N = \langle g_1, \dots, g_t \rangle_{\mathfrak{D}}.$$

- 2 Compute a filtered Gröbner basis $G = \{h_1, \dots, h_q\}$ of N .
- 3 Form

$$\text{gr}(N) = \langle \text{in}_{\text{ord}}(h_1), \dots, \text{in}_{\text{ord}}(h_q) \rangle_S \subset \text{gr}(F).$$

- 4 Build the bigraded module

$$R(M) = \bigoplus_{r \geq s \geq 0} (\text{gr}(F)_{r,s} / \text{gr}(N)_{r,s}) z^r s^s t^s.$$

- 5 Compute the bigraded Hilbert polynomial of $R(M)$.

Output

The bivariate Δ -dimension polynomial $P_M(r, s)$.

Example: two modules with the same univariate invariant

Let K have two commuting derivations ∂_1, ∂_2 , and let

$$F = \mathfrak{D}e_1 \oplus \mathfrak{D}e_2.$$

Fix integers $a > b \geq 1$. For the first module $M = F/N$, let N be generated by

$$g_1 = \partial_1^a e_1 + \partial_2^{a-b} e_2, \quad g_2 = \partial_1^{a+b} e_2 + \partial_2^{a-b} e_2.$$

A symbol-level Hilbert computation gives the univariate dimension polynomial

$$\phi_M(t) = (2a + b)t + \text{constant}.$$

Example: bivariate polynomial for M

After homogenization and passage to symbols, the window relations give

$$P_M(r, s) = (2a + b)r - (2a - 2b)s + \text{constant}.$$

Interpretation

The coefficient of r reflects the classical univariate leading growth. The coefficient of s records additional information from the lower boundary of the window.

A second module M'

Let $M' = F/N'$, where N' is generated by

$$g'_1 = \partial_1^a e_1 + \partial_2^a e_2, \quad g'_2 = \partial_1^{a+b} e_2.$$

The classical univariate polynomial has the same leading invariant:

$$\phi_{M'}(t) = (2a + b)t + \text{constant}.$$

But the window correction is different:

$$P_{M'}(r, s) = (2a + b)r - (2a)s + \text{constant}.$$

Comparison

The two modules have the same invariant detected by the classical univariate theory:

$\phi_M(t), \phi_{M'}(t)$ have the same degree and leading coefficient.

But their bivariate leading parts differ:

$$P_M(r, s) = (2a + b)r - (2a - 2b)s + \dots,$$

$$P_{M'}(r, s) = (2a + b)r - (2a)s + \dots.$$

Conclusion

Since the coefficients in the top-degree part are intrinsic,

$$M \not\cong M'.$$

The bivariate polynomial gives an effective nonisomorphism certificate.

Differential field extensions

Let $L = K\langle\eta_1, \dots, \eta_n\rangle$, the Δ -field extension of K generated by a set $\eta = \{\eta_1, \dots, \eta_n\}$.

The classical Kolchin's theorem states that there is a polynomial $\omega_{\eta|K}(r) \in \mathbb{Q}[r]$ such that

$$\omega_{\eta|K}(r) = \text{tr. deg}_K(\partial^\alpha \eta_i \mid |\alpha| \leq r, i = 1, \dots, n)$$

for all sufficiently large $r \in \mathbb{N}$.

Furthermore, the degree and leading coefficient of $\omega_{\eta|K}$ are independent of the choice of the set of Δ -generators η .

For a Δ -field extension $L = K\langle\eta_1, \dots, \eta_n\rangle$, define order-window fields

$$L_{r,s} = K(\partial^\alpha \eta_i \mid s \leq |\alpha| \leq r, i = 1, \dots, n).$$

Using the ω -module of Kähler differentials $\Omega_{L/K}$, the module construction gives a polynomial

$$P_{\eta|K}(r, s) \in \mathbb{Q}[r, s]$$

such that

$$\text{tr. deg}_K L_{r,s} = P_{\eta|K}(r, s)$$

for sufficiently large $r > s$.

Furthermore, the coefficients of all monomials $r^i s^j$ with $i + j = \deg P_{\eta|K}$ do not depend on the set of Δ -generators η of the extension L/K .

Let $R = K\{y_1, \dots, y_n\}$ be an algebra of Δ -polynomials over K . (As a ring, $R = K[\{\partial^\alpha y_i \mid \alpha \in \mathbb{N}^m, 1 \leq i \leq n\}]$).

The Δ -structure on R is obtained by extension of the action of ∂_i to R via $\partial_i(\partial_j y_k) = (\partial_i \partial_j) y_k$.) A system of algebraic differential (Δ -) equations over K is a system the form

$$f_i(y_1, \dots, y_n) = 0 \quad (i \in I), \quad (1)$$

where $f_i \in R$. By a solution we mean an n -tuple $(\eta_1, \dots, \eta_n)$ with coordinates in some Δ -overfield of K that annuls all f_i .

System (1) is said to be *prime* if the Δ -ideal $\mathcal{P}$ generated by the set $\{f_i\}_{i \in I}$ in R is prime. (As an ideal, $\mathcal{P}$ is generated by the set $\{\partial^\alpha f_i \mid \alpha \in \mathbb{N}^m, i \in I\}$.)

In this case, let $L = \text{q. f.}(R/\mathcal{P})$, the field of fractions of $R/\mathcal{P}$.

Then $L = K\langle \eta_1, \dots, \eta_n \rangle$ where η_i is the canonical image of y_i in $R/\mathcal{P}$.

In this case the (univariate) Kolchin Δ -dimension polynomial $\omega_{\eta|K}(r)$ of L/K , expresses the **Einstein's strength** of system (1), an important characteristic of a system of PDEs governing a physical field.

If C_r is the set of all coefficients of order $\leq r$ in Taylor expansions of solutions of the system, then the strength reflects the number of algebraically independent coefficients in C_r .

In this situation, the bivariate Δ -dimension polynomial $P_{\eta|K}(r, s)$ gives the number of algebraically independent coefficients in the set $C_{r,s}$ ($s \leq r$) of all Taylor coefficients whose order lies between s and r , so it reflects a refinement of the concept of strength.

Another application of bivariate Δ -dimension polynomials associated with prime systems of algebraic PDEs concerns the equivalence of such systems.

Two systems are equivalent if the corresponding differential field extensions L/K and L'/K are K -isomorphic.

The results about Δ -dimension polynomials show that if $P_L(r, s)$ and $P_{L'}(r, s)$ are bivariate Δ -dimension polynomials of these extensions (associated with some systems of Δ -generators) and

$$P_L^{\text{top}}(r, s) \neq P_{L'}^{\text{top}}(r, s),$$

then the corresponding differential field extensions are not isomorphic over K .

Practical meaning

Bivariate dimension polynomials can distinguish systems that have the same classical univariate invariants.

Thanks!

beamer-tu-l

beamer-ur-log