

Symbolic Integration in Weierstrass-like Extensions

Wenqiao Li

AMSS, Chinese Academy of Sciences

joint work with
Shaoshi Chen, Manuel Kauers, Xiuyun Li and David Masser

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Outline

- Hermite Reduction
- Weierstrass-like Extensions
- Special Poles

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$$\Rightarrow \int f \, dx = \frac{-3x}{(x^2 + 2)^2} + \frac{x - 3}{2(x^2 + 2)} + \int \frac{1}{2(x^2 + 2)} \, dx$$

Algebraic case

Example

Let y be algebraic over $\mathbb{Q}(x)$ with $y^3 - xy + 1 = 0$ and $f = \frac{-2x^2 + (6x^3 - 63)y + 3xy^2}{(4x^3 - 27)^2}$. Compute $\int f \, dx$.

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integral basis $\Rightarrow$ consistence.

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Weierstrass-like extensions

A Weierstrass $\wp$ function satisfies $\wp'(x)^2 = 4\wp^3(x) - g_2\wp(x) - g_3$ with $g_2, g_3 \in \mathbb{C}$ and $g_2^3 - 27g_3^2 \neq 0$.

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Problem

Given $f \in \mathbb{C}(x, \wp, \wp')$, determine whether $\exists g \in \mathbb{C}(x, \wp, \wp')$ s.t. $f = g'$, and compute such g if there exists one.

Weierstrass-like extensions

A transcendental element t over $\mathbb{C}(x)$ satisfies $m(t, t') = 0$ where $m(X, Y) \in \mathbb{C}(x)[X, Y]$ is absolutely irreducible and monic w.r.t. Y .

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An irreducible $p \in k[t]$ is **special** if it admits special points as its roots.

Otherwise, p is **normal**.

Field of algebraic functions

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$$p \text{ special} \iff \exists \text{ place } P \text{ lying above } p \text{ s.t. } \nu_P(p') > 0.$$

Such P is called a **special place**.

Hermite reduction in Weierstrass-like

Assume that $f = \frac{\sum_{i=1}^n f_i w_i}{D}$ with $\gcd(f_1, \dots, f_n, D) = 1$.

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$D_N = uv^{s+1}$, find $a_i \in k[t]$ s.t. $\frac{\sum_{i=1}^n b_i w_i}{uv^{s+1}} - \left(\frac{\sum_{i=1}^n a_i w_i}{v^s} \right)' = \frac{\dots}{\tilde{u}v^s}$

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$\{v^{s+1}(v^{-s}w_i)'\}_{i=1}^n$ forms a local integral basis at v

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$\{v^{s+1}(v^{-s}w_i)'\}_{i=1}^n$ forms a local integral basis at v

$\Uparrow$

Theorem + order comparison

Hermite reduction in Weierstrass-like

Assume that $f = \frac{\sum_{i=1}^n f_i w_i}{D}$ with $\gcd(f_1, \dots, f_n, D) = 1$.

Splitting factorization: $D = D_N D_S$

Canonical representation: $f = \underbrace{\frac{\sum_{i=1}^n b_i w_i}{D_N}}_{\mathcal{N}(f)} + \underbrace{\frac{\sum_{i=1}^n r_i w_i}{D_S}}_{\mathcal{S}(f)}, \deg_t b_i < \deg_t D_N$

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Theorem + order comparison

For $\tilde{u}$: (i) multiplicities $\leq s$, (ii) one can remove special factors.

Hermite reduction in Weierstrass-like Extensions

$$(w_1, \dots, w_n)' e = (w_1, \dots, w_n) \cdot M,$$

where $e \in k[t]$, $M \in k[t]^{n \times n}$ and $\gcd(e, M_{1,1}, \dots, M_{n,n}) = 1$.

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Theorem

Let $K = k(t, t')$ and $f \in K$. Then $\exists g \in K$,

$$\blacktriangleright f_0 = \frac{\sum_{i=1}^n r_i w_i}{e_N} \text{ with } r_i \in k[t] \text{ and}$$

$$\blacktriangleright h = \frac{\sum_{i=1}^n h_i w_i}{D_*} \text{ with } \deg h_i < \deg D_*,$$

D_* squarefree, no special factors and coprime with e , s.t.

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D_* squarefree, no special factors and coprime with e , s.t.

$$\mathcal{N}(f) = g' + f_0 + h.$$

Moreover, h is unique, and $h = 0$ if $f \in K'$.

Outline

- Hermite Reduction
- Weierstrass-like Extensions
- Special Poles

Assume that $(t')^2 = q(t)$ where $q(t) \in k[t]$ squarefree with $\deg_t q \geq 3$.

Back to Weierstrass

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Every special point of $k(t, t')$ is a constant in $\bar{k}$.

The hypothesis holds for $\mathbb{C}(x, y, y')$.

Special poles

$v \in k[t]$ special, $a, b \in k[t]$ with $\gcd(v, a, b) = 1$, $s \in \mathbb{Z}^+$, $f = \frac{a + bt'}{v^s}$.

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Complete reduction

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Let $K = k(t, t')$ and $f \in K$. Then $\exists g \in K$,

$$\blacktriangleright h = \frac{h_1 + h_2 t'}{D_*} \quad (\text{Hermite}),$$

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We say η a **polynomial remainder** of f .

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$$k = \mathbb{C}(x), \wp'(x)^2 = 4\wp(x)^3 - 4 \text{ and } t = \wp(x).$$

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Then t has no elementary integrals over $k(t, t')$.

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$$k = \mathbb{C}(x), \varphi'(x)^2 = 4\varphi(x)^3 - 4 \text{ and } t = \varphi(x).$$

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Then $f = \left(\frac{xt'}{t - 1} - \frac{xt'}{t} + x^2 + xt\right)' + \frac{xt'}{t - 1} - t$.

Thank You!