

# LCLM Decomposition of Differential Operators in Positive Characteristic

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Differential operators in  $k(t)\langle\partial\rangle = \{a_n(t)\partial^n + \cdots + a_1(t)\partial + a_0(t)\}$ .

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- Can I find  $L_1$  and  $L_2$  “**with no overlap**” st  $L$  is the smallest left multiple of  $L_1$  and  $L_2$ .

# Solution spaces and LCLM decompositions

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- $\text{GCRD}(L_1, L_2) = 1$ .
- $\text{ord}(L) = \text{ord}(L_1) + \text{ord}(L_2)$ .
- $\mathcal{D}_L \simeq \mathcal{D}_{L_1} \oplus \mathcal{D}_{L_2}$

# In characteristic 0: **Singer's** Eigenring method

$$\mathcal{D}_{L_1} \oplus \mathcal{D}_{L_2} \xrightarrow{\text{Id} \oplus 0} \mathcal{D}_{L_1} \oplus \mathcal{D}_{L_2}$$

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- Compute  $\chi(\pi)$ .
- Use the kernel decomposition lemma.

# State of the art

- T. Cluzeau, factorisation of differential systems in characteristic  $p$ , ISSAC 2003.
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In char  $p$ :  $\chi(\pi) \in \mathbb{F}_p(t^p)[Y]$ .

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- M. van der Put. Differential Equations in Characteristic  $p$ , 1995.

# Overview

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- II) Compute a representative  $L_*$  with known LCLM-decomposition.
- III) Compute an isomorphism  $\mathcal{D}_{L_*} \simeq \mathcal{D}_L$ .

# Contribution

## Theorem

*There exists a Las Vegas type algorithm that given an operator  $L \in \mathbb{F}_{p^n}[t]\langle \partial \rangle$  of order  $r$  with coefficients of degree  $d$  satisfying some separability hypothesis, returns a complete LCLM-decomposition of  $L$  in polynomial time in  $r$  and  $d$ , and at most quasiquadratic time in  $p$ .*

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## Remark

Experiments suggest quasilinear in  $p$ .

# $p$ -curvature

## Definition

Let  $L \in \mathbb{F}_q(t)\langle\partial\rangle$ .

$$\begin{array}{ccc} \psi_p^L : & \mathcal{D}_L & \rightarrow \mathcal{D}_L \\ & M & \mapsto \partial^p M \pmod{L} \end{array}$$

## Facts

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- **Grothendieck-Katz:**  $\dim_{\overline{\mathbb{F}_q(t)}}(L) = \dim \ker \psi_p^L$ .

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- **(van der Put)**  $\mathcal{D}_L \simeq \mathcal{D}_{L^*}$  iff  $\psi_p^L \simeq \psi_p^{L^*}$ .

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**Consequence:** We can assume that  $L$  divides some  $N(t^p, \partial^p)^\nu$  for  $N \in \mathbb{F}_q[X, Y]$  irreducible and  $\nu \in \mathbb{N}$ .

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**Not constructive:**  $\psi_p^L = G \begin{pmatrix} C_{N^{\nu_1}} & & & \\ & C_{N^{\nu_2}} & & \\ & & \ddots & \\ & & & C_{N^{\nu_k}} \end{pmatrix} G^{-1}.$

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The blocks are **not stable by derivation**.

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$$\mathcal{D}_L \simeq \mathcal{D}_{\text{LCLM}_{i=1}^k L_i}.$$

# Parametrizing irreducible divisors

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## Theorem (van der Put)

*If  $N(t^p, \partial^p)$  is reducible, the irreducible divisors of  $N(t^p, \partial^p)$  are parametrized by the solutions of*

$$f^{(p-1)} + f^p = y_N$$

*in  $K_N$ . If  $N(t^p, \partial^p)$  is irreducible then this equation has no solution in  $K_N$ .*

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## Lemma (Pagès 2026)

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- Let  $y_N$  be a root of  $N(x^p, Y)$  in a separable closure of  $\mathbb{F}_q(t)$ .
- $C_N = \mathbb{F}_q(t^p)[y_N]$ ,  $K_N = \mathbb{F}_q(t)[y_N]$ .

## Lemma (Pagès 2026)

*Let  $\nu_1 \leq \dots \leq \nu_k \in \mathbb{N}$  with  $k \leq p$ . Let  $f \in K_N$  verify  $f^{(p-1)} + f^p = y_N$ . Set  $L_i$ : the smallest multiple of  $(x\partial - xf + i)^{\nu_i}$  in  $\mathbb{F}_q(t)\langle\partial\rangle$ .*

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- **no overlap**:  $\text{ord LCLM}_{i=1}^k L_i = \sum_{i=1}^k \text{ord } L_i$

## Theorem (van der Put)

Let  $L$  be a non trivial divisor of  $N(t^p, \partial^p)$  and  
 $\text{GCRD}(L, \partial^p - y_N) = \partial^m + l_{m-1}\partial^{m-1} + \cdots + l_0 \in K_N\langle\partial\rangle$ . Then

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## Theorem (Pagès 2024)

Let  $S$  be a set a places of  $K_N$  generating  $\text{Pic}(K_N)/p\text{Pic}(K_N)$ .  
 If  $N(t^p, \partial^p)$  is reducible then a solution of  $f^{(p-1)} + f^p = y_N$  lives in the  
 Riemann-Roch space

$$\mathcal{L}\left(\max\left(\sum_{\mathfrak{P} \in S} \mathfrak{P} + \text{Diff}(K_N) - 2(t)_{\infty}, p^{-1} \cdot (y_N)_{\infty}\right)\right)$$

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- Compute the LCLM of all factors

# Isomorphism

If  $L, L^* \in \mathbb{F}_q(t)\langle\partial\rangle$ , computing  $\mathcal{M}\text{or}_{\mathbb{F}_q(t)\langle\partial\rangle}(\mathcal{D}_{L^*}, \mathcal{D}_L)$  is a matter of solving  $L^*M \equiv 0 \pmod L$  with  $\text{ord}(M) < \text{ord } L$ .

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## Lemma

*If  $\varphi : \mathcal{D}_{L^*} \xrightarrow{\sim} \mathcal{D}_L$  and  $L_1^*, \dots, L_k^*$  is a LCLM-decomposition of  $L_*$  then  $(\text{GCRD}(\varphi(L_i^*), L))_i$  is a LCLM-decomposition of  $L$ .*

**Thank you for your attention.**